1. Function R is defined by $R(x) = \dfrac{2\,x^2 - 18}{(5\,x - 15)\cdot(2-x)}$.

   a) Explain why 3 is not in the domain of this function.
   Solution: Denominator term $5\,x - 15$ has value zero if $x = 3$.

   b) Factor and simplify the expression for $R(x)$.
   Solution: $R(x) \;=\; \dfrac{2\,x^2 - 18}{(5\,x - 15)\cdot(2-x)} \;=\; \dfrac{2\cdot(x-3)\cdot(x+3)}{5\cdot(x-3)\cdot(2-x)} \;=\; \dfrac{2\cdot(x+3)}{5\cdot(2-x)}$ if $x \neq 3$.

   c) Show work which computes $\lim_{x\to 3} R(x)$ or shows that limit does not exist.
   Solution: $\lim_{x\to 3} R(x) \;=\; \lim_{x\to 3} \dfrac{2\cdot(x+3)}{5\cdot(2-x)} \;=\; \dfrac{2\cdot(3+3)}{5\cdot(2-3)} \;=\; \dfrac{-12}{5}$

   d) What do the results of parts (b) & (c) imply about the graph of $y = R(x)$ for x in $[2,4]$?
   Solution: Graph of R has a "hole" at the point $\left(3, \dfrac{-12}{5}\right)$; that is a *removable disconinuity* for this function.

2. Consider the following statement.

   $$\textit{If } \lim_{x\to 5} f(x) \textit{ exists and if } f(5)\neq 0\textit{, then } \lim_{x\to 5} \frac{1}{f(x)} \textit{ exists.}$$

   Explain why that general statement is not correct. [Hint: ¿ Is there a function for which that statement is false?]

   Solution: Let g be your favorite continuous function which has value zero at $x = 5$, then define f to have $f(5) = 13$ and $f(x) = g(x)$ for every $x \neq 5$.

3. a) Explain why $\lim_{x\to\infty} \sin(x)$ does not exist.
   Solution: In every interval of length $2\,\pi$, the Sine function oscillates continuously between -1 and 1; there is no number L such that values of Sine get close AND stay close. [Example 7 on page 62 involves a similar situation.]

   b) Show $\lim_{x\to\infty} \dfrac{\sin(x)}{x}$ does exist and compute its value.
   Solution: $-1 \leq \sin(x) \leq 1$ implies $\dfrac{-1}{x} \leq \dfrac{\sin(x)}{x} \leq \dfrac{1}{x}$ for all positive x. Since $\lim_{x\to\infty} \dfrac{\pm 1}{x} = 0$, an intuitive "Squeeze Theorem" would imply $\dfrac{\sin(x)}{x}$ has the same limit.

4. $f(x) = \dfrac{x^2 + 2\,x - 3}{x^3 - 1}$ is not defined for $x = 1$. Is there a number c such that letting $f(1) = c$ would yield a revised function which would be continuous at $x = 1$? Explain your answer.

   Solution: $\dfrac{x^2 + 2\,x - 3}{x^3 - 1} \;=\; \dfrac{(x+3)\cdot(x-1)}{(x-1)\cdot(x^2+x+1)} \;=\; \dfrac{x+3}{x^2+x+1}$ if $x \neq 1$. Because $\dfrac{x+3}{x^2+x+1} = 4\cdot\dfrac{x+3}{(2\,x+1)^2+3}$ is defined for all x, letting $f(1) = \dfrac{1+3}{1^2+1+1} = \dfrac{4}{3}$ revises f so that it is continuous everywhere.

   Alternative using a fancier tool: The expression for f is indeterminate of type $\frac{0}{0}$ at $x = 1$. Careful use of l'Hôpital's Rule will yield the same limit value and same conclusion.

5. Find a value of parameter A for which the following function is continuous at 5. Explain why your choice of A "works".

   $$f(x) \;=\; \begin{cases} A + x & \text{if } \; x \leq 5 \\ A\cdot x & \text{if } \; x > 5 \end{cases}$$

   Solution: Pick A so that both one-sided limit values at 5 have the same value as $f(5)$.

   $$\lim_{x\to 5^-} f(x) \;=\; \lim_{x\to 5^-} (A + x) \;=\; A + 5 \;=\; f(5)$$
   $$\lim_{x\to 5^+} f(x) \;=\; \lim_{x\to 5^-} (A\cdot x) \;=\; A\cdot 5$$

   Equation $A + 5 = A\cdot 5$ has solution $A = \frac{5}{4}$; with that value for A we have $\lim_{x\to 5} f(x) = f(5)$.

6. a) Use a **limit** to define the derivative function of $f(x) = x^2 - 3$.
   Solution: $f'(x) \;=\; \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} \;=\; \lim_{h\to 0} \dfrac{\big((x+h)^2 - 3\big) - \big(x^2 - 3\big)}{h}$

b) Use the previous **limit** to verify that $f'(2) = 4$.
Solution:
$$\lim_{h\to 0} \frac{\big((2+h)^2-3\big)-\big(2^2-3\big)}{h} = \lim_{h\to 0}\frac{(2+h)^2-2^2}{h} = \lim_{h\to 0}\frac{4+4h+h^2-4}{h}$$
$$= \lim_{h\to 0}\frac{4h+h^2}{h} = \lim_{h\to 0}\frac{(4+h)\cdot h}{h} = \lim_{h\to 0}(4+h) = 4+0 = 4$$

7. Let $g(x) = \sqrt{x+4}$. Use the limit definition of a derivative to verify that $g'(0) = \frac{1}{4}$.

Solution: A key step in this algebra uses $(A-B)\cdot(A+B) = A^2 - B^2$ with $A = \sqrt{h+4}$ and $B = 2$.
$$\begin{aligned} g'(0) &= \lim_{h\to 0}\frac{g(0+h)-g(0)}{h} = \lim_{h\to 0}\frac{\sqrt{(0+h)+4}-\sqrt{0+4}}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{h+4}-\sqrt{4}}{h} = \lim_{h\to 0}\frac{\sqrt{h+4}-2}{h}\cdot\frac{\sqrt{h+4}+2}{\sqrt{h+4}+2} = \lim_{h\to 0}\frac{\big(\sqrt{h+4}\big)^2-2^2}{h\cdot\big(\sqrt{h+4}+2\big)} \\ &= \lim_{h\to 0}\frac{(h+4)-4}{h\cdot\big(\sqrt{h+4}+2\big)} = \lim_{h\to 0}\frac{h}{h\cdot\big(\sqrt{h+4}+2\big)} = \lim_{h\to 0}\frac{h}{h}\cdot\frac{1}{\sqrt{h+4}+2} \\ &= \lim_{h\to 0}\frac{1}{\sqrt{h+4}+2} = \frac{1}{\sqrt{0+4}+2} = \frac{1}{2+2} \end{aligned}$$

8. The line with equation $3x - 4y = 7$ is tangent to the graph of $y = f(x)$ at the point with $x = -5$.

a) $f(-5) = -11/2$
b) $f'(-5) = 3/4$

Solution: The equation can be rewritten in the form $y = \frac{3x-7}{4} = \frac{3}{4}x - \frac{7}{4}$. Its graph goes through the point $\left(-5, \frac{3\cdot(-5)-7}{4}\right) = \left(-5, \frac{-11}{2}\right) = \big(-5, f(-5)\big)$ with slope $\frac{3}{4} = f'(-5)$.

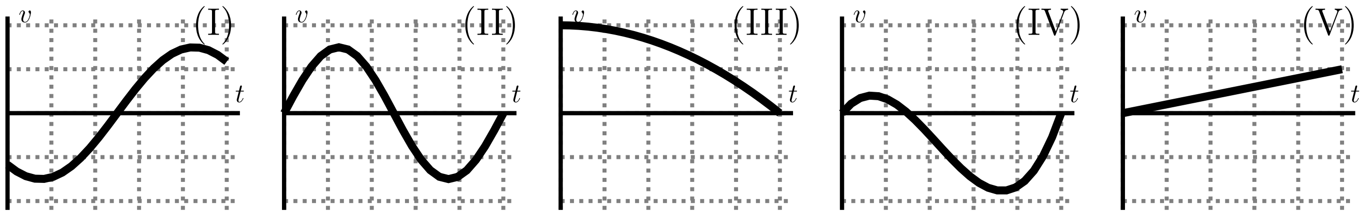
9. Use an appropriate local linear approximation to estimate the value of $\sqrt[3]{7.98}$

Solution: $g(x) = \sqrt[3]{x} = x^{1/3}$ has derivative $g'(x) = \frac{1}{3\,x^{2/3}}$. The local linear approximation of g at $8 = 2^3$ is
$$L(x) = g(8) + g'(8)\cdot(x-8) = 8^{1/3} + \frac{1}{3\cdot 8^{2/3}}\cdot(x-8) = 8 + \frac{1}{12}\cdot(x-8)$$
This yields $\sqrt[3]{7.98} = g(7.98) \approx L(7.98) = 2 + \frac{1}{12}\cdot(7.98-8) = 2 + \frac{-0.02}{12} = 2 - \frac{1}{600}$ [a good non-calculator answer]

10. An arrow is shot upwards. If the arrow hits the ground 20 seconds later, what was its initial velocity?

Solution: Let $h(t)$ be the arrow's height t seconds after it was "launched". Use simple model $h(0) = 0$ [i.e., shot from ground-level] and gravity provides the only force on this arrow during its flight [i.e., ignore air resistance, etc.]. Then $h'(0) = v_0$ is the initial velocity and $h''(t) = g$ is the constant (negative) acceleration due to gravity. Those conditions imply $v(t) = h'(t) = v_0 + g\,t$ and $h(t) = h(0) + v_0\,t + g\,\frac{t^2}{2} = 0 + v_0\,t + \frac{g}{2}\,t^2 = t\cdot\left(v_0 + \frac{g}{2}t\right)$. If the arrow returns to ground at $t = 20$ seconds, then $0 = h(20)$ implies $v_0 + \frac{g}{2}\cdot 20 = 0$ which is equivalent to $v_0 = \frac{-g}{2}\cdot 20 = -10\,g$. Using two popular estimates for g, we find the initial velocity, v_0, to be $98\frac{\text{m}}{\text{s}} \approx 320\frac{\text{ft}}{\text{s}}$.

11. These graphs present velocity, $v(t)$, for several particles moving along the x-axis for times $0 \le t \le 5$. [All figures use the same scales.]



For each of the following description of a motion, identify which graph(s) show that behavior.

a) has a constant acceleration
Solution: $a(t) = v'(t)$ is constant only for figure V.

b) ends up farthest to the left of where it started
Solution: Displacement, $\int_0^5 v(t)\, dt$, is positive for III & V, approximately zero for I & II; it's negative only for IV

c) ends up farthest to the right from its starting point
Solution: Positive displacent (to the right) for III is more that two times the displacement for V.

d) experiences the greatest initial acceleration
Solution: Initial acceleration, $v'(0)$, is negative for I , nearly zero for III, 1/5 for V; tangents at origin for II and IV have positive slope, but $a(0) = v'(0)$ is larger for II.

e) has the greatest average velocity
Solution: $\dfrac{1}{5-0}\displaystyle\int_0^5 v(t)\, dt = \dfrac{\text{displacement}}{\text{duration}}$ is largest for III.

f) has the greatest average acceleration
Solution: $\dfrac{1}{5-0}\displaystyle\int_0^5 v'(t)\, dt = \dfrac{v(5)-v(0)}{5}$ is largest for I.

12. Suppose f and g are differentiable functions with values shown in these tables.

x	$f(x)$	$f'(x)$
2	3	5
4	-7	6

x	$g(x)$	$g'(x)$
2	4	-2
4	3	9

For each of the following parts, show how to use those table values to compute the specified derivative OR explain what extra information would be needed to finish that computation.

a) If $A(x) = f(x) - g(x)$, then $A'(4) = f'(4) - g'(4) = 6 - 9 = -3$
b) If $B(x) = f(x) \cdot g(x)$, then $B'(2) = f'(2) \cdot g(2) + f(2) \cdot g'(2) = 5 \cdot 4 + 3 \cdot (-2) = 20 - 6 = 14$
c) If $C(x) = \dfrac{g(x)}{f(x)}$, then $C'(4) = \dfrac{g'(4) \cdot f(4) - g(4) \cdot f'(4)}{f(4)^2} = \dfrac{9 \cdot (-7) - 3 \cdot 6}{(-7)^2} = \dfrac{-63-18}{49} = \dfrac{-81}{49}$
d) If $D(x) = f\big(g(x)\big)$, then $D'(2) = f'\big(g(2)\big) \cdot g'(2) = f'(4) \cdot g'(2) = 6 \cdot (-2) = -12$

13. Domain of function $f(x) = \ln\Big(\big|3\,x\big|\Big)$ includes all non-zero real numbers. Show work which finds an expression for $f'(x)$.

Hint: treat the case $x > 0$ first, then consider the case $x < 0$ separately. Simplify each result, then simplify further.

Solution: If $x > 0$, then $f(x) = \ln(3\,x) = \ln(3) + \ln(x)$ implies $f'(x) = \dfrac{1}{x}$. On the other hand, if $x < 0$, then $f(x) = \ln(-3\,x)$ implies $f'(x) = \dfrac{1}{-3\,x} \cdot (-3) = \dfrac{1}{x}$. Because both cases yield the same simple expression, $f'(x) = \dfrac{1}{x}$ for all non-zero values of x.

14. Differentiate some expressions involving Sine and Cosine several times.

a) Find the third derivative of $f(x) = \sin(7 + x)$.
Solution: $\sin' = \cos$, $\sin'' = \cos' = -\sin$, and $\sin''' = -\sin' = -\cos$; therefore $f'''(x) = -\cos(7 + x)$.

b) Find the fifth derivative of $g(x) = \cos(2\,x)$.
Solution: Fourth derivative of Cosine is Cosine; its fifth derivative, $\cos^{(5)}$, is $\cos' = -\sin$. Differentiating g five times will also need to apply the Chain Rule five times. Therefore $g^{(5)}(x) = \big(-\sin(2\,x)\big) \cdot 2^5 = -32\,\sin(2\,x)$.

c) Explain why repeatedly differentiating $\sin(x)$ will never produce $-2\cos(x)$.
Solution: The various derivatives of $\sin(x)$ are $\pm\cos(x)$ and $\pm\sin(x)$; each has amplitude 1 but $-2\cos(x)$ has amplitude 2.

15. Consider the equation $e^y - 2x^3 y + 4 = 5x$.

a) Find an expression for $\dfrac{dy}{dx}$.
Solution: Differentiate the equation (with respect to x), then solve for $\frac{dy}{dx}$.

$$\begin{aligned} 5 = \frac{d}{dx}(5x) &= \frac{d}{dx}\left(e^y - 2x^3 y + 4\right) \\ &= \frac{d}{dx}\left(e^y\right) - 2\cdot\frac{d}{dx}\left(x^3\cdot y\right) + 0 \\ &= e^y\cdot\frac{dy}{dx} - 2\cdot\left(3x^2\cdot y + x^3\cdot\frac{dy}{dx}\right) \end{aligned}$$

is equivalent to

$$5 + 2\cdot 3x^2\cdot y = e^y\cdot\frac{dy}{dx} - 2\cdot x^3\cdot\frac{dy}{dx} = \left(e^y - 2x^3\right)\cdot\frac{dy}{dx}$$

with solution

$$\frac{dy}{dx} = \frac{5 + 6x^2 y}{e^y - 2x^3}$$

b) What condition must x and y satisfy so that the corresponding tangent line is horizontal?
Solution: Intersect the original curve with the graph of $y = -5/(6x^2)$ [equivalent to $5 + 6x^2 y = 0$]; for example, $(1.3956, -0.4279)$ is near such a point.

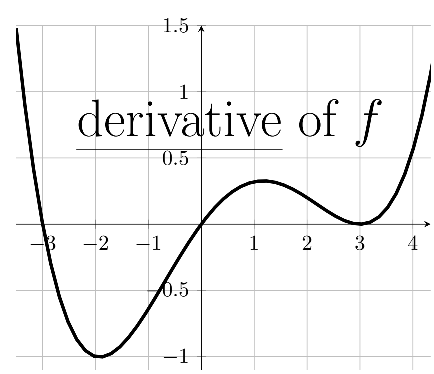
c) What condition must x and y satisfy so that the corresponding tangent line is vertical?
Solution: Intersect the original curve with graph of $y = \ln(2x^3)$ [equivalent to $e^y - 2x^3 = 0$]; for example, $(0.9550, 0.5551)$ is near such a point.

16. Suppose g is a differentiable function such that $-1 \le g'(x) \le 3$ for all x in $[0, 12]$. Also suppose $g(2) = 5$.

a) What is the smallest possible value for $g(7)$? Explain your reasoning.
Solution: If $g'(x)$ was equal to the lower bound of -1 for $2 \le x \le 7$, then $g(7)$ would be $5 + (-1)\cdot(7-2) = 0$.

b) What is the largest possible value for $g(7)$? Explain your reasoning.
Solution: If $g'(x)$ was equal to the upper bound of 3 for $2 \le x \le 7$, then $g(7)$ would be $5 + 3\cdot(7-2) = 20$.

17. Examine this graph of the derivative of some function f.



a) Determine where f is decreasing.
Solution: f is decreasing on intervals where its derivative has negative values; for this graph of f', the answer is open interval $(-3, 0)$.

b) Locate the x values for the local extrema of f.
Solution: f has a local maximum at -3 where sign of f' changes from positive to negative; f has a local minimum at 0 where sign of f' changes from negative to positive.

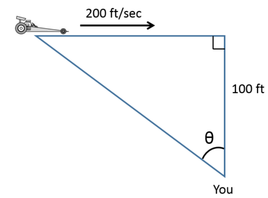
c) Determine where f is concave up.
Solution: f is concave up on intervals where f' is increasing; for this graph of f', that seems to be [approximately] $(-2, 1)$ and $(3, \infty)$.

d) Locate the x values for any inflection point of f.
Solution: Graph of f changes from concave down to concave up near -2, from up to down near 1, and from down to up at 3.

18. Find the global extrema of the function $f(x) = 2x - e^x$ on the interval $[-1, 2]$.

Solution: $f'(x) = 2 - e^x$ is positive on $[-1, \ln(2))$, zero at $\ln(2)$, and negative on $(\ln(2), 2]$. A generalized First Derivative Test implies $f\big(\ln(2)\big) = 2\ln(2) - e^{\ln(2)} = 2\ln(2) - 2 \approx -0.61$ is the global maximum on interval $[-1, 2]$. At the ends of the interval, we have $f(-1) = 2\cdot(-1) - e^{-1} \approx -2.37$ while $f(2) = 2\cdot 2 - e^2 \approx -3.39$ is smaller; thus $f(2)$ is the global minimum.

19. Examine this figure. You are taking a video of a racecar from a position that is 100 feet from the racetrack. The racecar is moving at constant rate of 200 feet-per-second. How fast is your viewing angle, θ in this figure, changing at the moment when the car is directly in front of you?



Solution: Let $x(\theta)$ be distance of the racecar from the spot on the racetrack that is directly opposite you, i.e., at the point marked as a right angle. Geometry implies $x(\theta) = 100\,\tan(\theta)$ and car's velocity implies $\dfrac{dx}{dt} = -\,200\,\dfrac{\text{ft}}{\text{s}}$. Therefore

$$-200 \;=\; \frac{d}{dt}x(\theta) \;=\; \frac{d}{dt}\Big(100\,\tan(\theta)\Big) \;=\; 100\,\frac{1}{\cos(\theta)^2}\cdot\frac{d\theta}{dt}$$

The racecar is directly in front of you when $\theta = 0$; at that moment

$$\left.\frac{d\theta}{dt}\right|_{\theta=0} \;=\; (-200)\cdot\left.\frac{\cos(\theta)^2}{100}\right|_{\theta=0} \;=\; \frac{-200}{100}\cdot\cos(0)^2 \;=\; -2\cdot 1^2 \;=\; -2\,\frac{\text{radians}}{\text{s}}$$

Alternative Solution: The relation $x = 100\,\tan(\theta)$ implies $\theta = \arctan\left(\dfrac{x}{100}\right)$. Therefore

$$\frac{d\theta}{dt} \;=\; \arctan'\left(\frac{x}{100}\right)\cdot\frac{d}{dt}\left(\frac{x}{100}\right) \;=\; \frac{1}{1+(x/100)^2}\cdot\frac{1}{100}\cdot\frac{dx}{dt} \;=\; \frac{1}{1+(x/100)^2}\cdot\frac{1}{100\text{ ft}}\cdot\left(-\,200\,\frac{\text{ft}}{\text{s}}\right)$$

Finish by evaluating that at the moment when $x = 0$ feet.

20. Starting at time $t = 0$ hours, water leaks out of a tank at rate $r(t)$, measured in gallons-per-hour. This table has some values for $r(t)$.

t [hr]	0.0	0.5	1.0	1.5	2.0	2.5	3.0
$r(t)$ [gal/hr]	0	6	11	15	18	20	21

a) What is the meaning of the quantity $\displaystyle\int_1^2 r(t)\,dt$? Write your answer in terms relevant to this situation AND make it understandable by someone who does not know any calculus; mention any units that are appropriate.
Solution: The notation designates accumulation of the leaking rate r during the time between hours 1 and 2; in other words, it computes the total amount of water which leaked from the tank in that time interval.

b) Write a three-term Riemann sum to estimate $\displaystyle\int_0^3 r(t)\,dt$ [this is not the definite integral given in part (a)]. Give your answer as an expression in terms of numbers, but you do not need to do the arithmetic.
Solution: Among the variety of reasonable expressions, here are three which use $\Delta t = 1\,\text{hr}$.

$$\begin{aligned}\text{Left} &= (0\times\Delta t)+(11\times\Delta t)+(18\times\Delta t) \;=\; 0+11+18\\ \text{Mid} &= (6\times\Delta t)+(15\times\Delta t)+(20\times\Delta t) \;=\; 6+15+20\\ \text{Right} &= (11\times\Delta t)+(18\times\Delta t)+(21\times\Delta t) \;=\; 11+18+21\end{aligned}$$

21. Evaluate the following definite integrals and antiderivative.

a) $\displaystyle\int_{-1}^{3}(w+1)\cdot(w-1)\cdot(w-3)\,dw \;=\; \int_{-1}^{3} w^3-3w^2-w+3\,dw = \frac{w^4}{4}-w^3-\frac{w^2}{2}+3w\Bigg|_{-1}^{3} = \frac{-9}{4}-\left(\frac{-9}{4}\right) = 0$

b) $\displaystyle\int_1^{e^3}\frac{5}{x}-6x^2+\sqrt{x}\,dx \;=\; 5\,\ln(x)-2\,x^3+\frac{2}{3}\,x^{3/2}\Bigg|_1^{e^3} \;=\; \left(15-2\,e^9+\frac{2}{3}\,e^{9/2}\right)-\left(\frac{-4}{3}\right)$

c) $\displaystyle\int 5\,e^t+\sin(t)\,dt \;=\; 5\,e^t-\cos(t)+C$

22. Shade a region whose area is computed by $\displaystyle\int_{-5}^{5}\sqrt{25-x^2}\,dx$; then use geometry to evaluate the definite integral.

Solution: A semicircle with radius $r = 5$ has area $\dfrac{1}{2}\left(\pi\,r^2\right) \;=\; \dfrac{1}{2}\cdot\pi\cdot 5^2$