3.10:18 [4=2+2 pts] A trip started at 11:44pm and ended at 1:12am took $60 + 16 + 12 = 88$ minutes. Since the trip's distance was 116 miles, Dominic's average speed was $\dfrac{116}{88} = \dfrac{29}{22}\dfrac{miles}{minute} = \dfrac{29 \times 60}{22}\dfrac{miles}{hour} \approx 79.1$ miles-per-hour. The Mean Value Theorem (for differentiable functions) implies there must have been a moment when Dominic's instantaneous speed equaled that average speed. Because speed limit on that highway never exceeded 75 miles-per-hour, that means **Dominic drove faster than the speed limit** at some time during his trip.

4.1:22 [6=2+4 pts] Parameter a is a non-zero constant and $f(x) = \dfrac{a}{x^2} + x = a \cdot x^{-2} + x$.

a) $f'(x) = a \cdot (-2) \cdot x^{(-2)-1} + 1 = \dfrac{-2\,a}{x^3} + 1$. The only solution of $f'(x) = 0$ is $x = (2\,a)^{1/3} = \sqrt[3]{2\,a}$. That is the only critical point for f. [0 is not a critical point for f because 0 is not in domain of f.]

b) $f''(x) = (-2\,a) \cdot (-3) \cdot x^{(-3)-1} = \dfrac{6\,a}{x^4}$. Although sign of $c = \sqrt[3]{2\,a}$ matches the sign of a, the value of $c^4 = \left(\sqrt[3]{2\,a}\right)^4$ is always positive; that implies sign of $f''(c) = \dfrac{6\,a}{c^4}$ also matches sign of a.

- $a > 0$ implies value of f'' at the critical point is positive; the Second Derivative Test implies f has a local minimum at $c = \sqrt[3]{2\,a}$.
- $a < 0$ implies value of f'' at the critical point is negative; the Second Derivative Test implies f has a local maximum at $c = \sqrt[3]{2\,a}$.

4.2:30 [6=4+2 pts] The patient's temperature change T due to dose D of a drug is

$$T = g(D) = \left(\frac{C}{2} - \frac{D}{3}\right) \cdot D^2$$

where C is a positive constant.

a) $\dfrac{dT}{dD} = g'(D) = \dfrac{d}{dD}\left(\dfrac{C}{2} \cdot D^2 - \dfrac{1}{3}D^3\right) = \dfrac{C}{2} \cdot (2\,D) - \dfrac{1}{3}\left(3\,D^2\right) = C \cdot D - D^2 = (C - D) \cdot D$

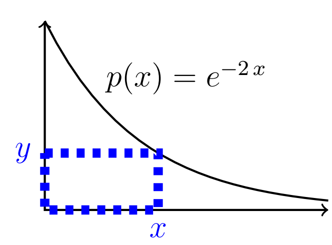
The only critical points (zeros of dT/dD) for this relation are $D = 0$ and $D = C$. The product $(C - D) \cdot D$ is positive only if D is between 0 and C, it is negative elsewhere. Because $0 < C$, the First Derivative Test implies T has a local minimum at $D = 0$ and a local maximum at $D = C$.

Note: $g(0) = (\text{stuff}) \cdot 0^2 = 0$ while $g(C) = \left(\dfrac{C}{2} - \dfrac{C}{3}\right) \cdot C^2 = \dfrac{C^3}{6}$ is positive only if $C > 0$. If C were a negative constant, then T would have a local minimum at $D = C$.

b) If a patient's sensitivity S to amount D of this drug is defined as $S = f(D) = \dfrac{dT}{dD}$, then part (a) implies $S = g'(D) = C \cdot D - D^2$. Therefore $\dfrac{dS}{dD} = f'(D) = g''(D) = \dfrac{d^2T}{dD^2} = C - 2\,D$ so the only critical point for S is $D = \dfrac{C}{2}$. Because dS/dD changes from positive to negative at that critical point, the First Derivative Test implies sensitivity S is **maximized** at $D = C/2$.

4.3:14 [6=3+3 pts] The dashed rectangle has

- one side on the x-axis,
- one side on the y-axis,
- one vertex at the origin, and
- the opposite vertex on graph of $p(x) = e^{-2\,x}$ for $x \geq 0$.



a) The rectangle has area $A = x \cdot y = x \cdot p(x) = x \cdot e^{-2\,x}$. Therefore

$$\frac{dA}{dx} = 1 \cdot e^{-2\,x} + x \cdot e^{-2\,x} \cdot (-2) = (1 - 2\,x) \cdot e^{-2\,x}$$

Notice that $\dfrac{dA}{dx}$ is positive on $\left[0, \dfrac{1}{2}\right)$, zero at $\dfrac{1}{2}$, and negative on $\left(\dfrac{1}{2}, \infty\right)$. Therefore A has its global maximum of $\dfrac{1}{2} \cdot e^{-2\cdot(1/2)} = \dfrac{1}{2\,e}$ at $x = \dfrac{1}{2}$.

b) The rectangle has perimeter $P = 2\,x + 2\,y = 2\,x + 2\,e^{-2\,x}$. Therefore
$$\frac{dP}{dx} = 2 \cdot 1 + 2 \cdot e^{-2\,x} \cdot (-2) = 2 - 4\,e^{-2\,x}$$
The critical x-value satisfies $e^{-2\,x} = \dfrac{2}{4}$ which is equivalent to $-2\,x = \ln\left[\dfrac{1}{2}\right]$ and $x = \dfrac{1}{-2} \cdot \ln\left[\dfrac{1}{2}\right] = \dfrac{1}{2}\ln(2)$. The second derivative is easy to compute and analyze.
$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left(2 - 4\,e^{-2\,x}\right) = -4 \cdot e^{-2\,x} \cdot (-2) = 8 \cdot e^{-2\,x}$$
is positive for all x. Therefore graph of $P = f(x)$ is concave up everywhere and the global minimum perimeter is
$$f\left(\frac{1}{2}\ln(2)\right) = 2 \cdot \left(\frac{1}{2}\ln(2)\right) + 2 \cdot e^{-2 \cdot \left(\frac{1}{2}\ln(2)\right)} = \ln(2) + 2 \cdot e^{-\ln(2)} = \ln(2) + 2 \cdot \frac{1}{2} = \ln(2) + 1$$

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remarks about points and partial credit

3.10:18 [4=2+2 pts]

- 2 points for computing elapsed time and average speed
- 2 points for citing the Mean Value Theorem after comparing that average speed with 75 mph speed limit

4.1:22 [6 = (1+1) + (2+2) pts]

a) 1 point for derivative, 1 point for critical point

(comment but don't penalize for a false claim that 0 is a critical point)

b) 1 point for $f\,''$ and 1 point for noting sign of $f\,''(x)$ matches sign of a;

then 1 point each for justifying ($a > 0$ implies local min) and ($a < 0$ implies local max)

Note: since sign for values of $f\,''(x)$ for any choice of x depends on just the sign of a, it is OK if the min/max classification is stated in terms of f being concave up everywhere if $a > 0$ and concave down if $a < 0$. (My solution cites the Second Derivative Test, a surgical tool, in a context where a blunt instrument, concavity everywhere, would suffice ;-)

4.2:30 [6=(2+2)+2 pts]

a) 2=1+1 points for basic stuff: 1 for the derivative, 1 for declaring T is maximal when $D = C$;

2=1+1 more for subtle stuff: 1 for mentioning critical point at $D = 0$ and **last point for recognizing that not getting a local min at $D = C$ requires C be positive** [for-whatever-it's-worth, the publisher's "solution" misses that last point]. On the other hand, don't deduct that last point if something like the following is mentioned.

In a clinical context, an actual dosage amount D would be positive.

(That observation is not the whole story, but it looks in the right direction.)

b) 1 for derivative (using a new name, S, for sensitivity is not needed),

1 for applying First Derivative Test to get conclusion about maximizing sensitivity.

4.3:14 [6=3+3 pts] In each part:

- 1 for derivative
- 1 for analysis of derivative to locate critical x-value
- 1 for finishing by computing the actual extreme value: $\dfrac{1}{2\,e}$ for part a and $\ln(2) + 1$ for part b

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Supplement to remark in solution for part (a) of 4.2:30. If c is constant, then product $(c - x) \cdot x$ changes sign at $x = c$ and $x = 0$. If $|x|$ is huge, then $(c - x) \cdot x \approx -x \cdot x < 0$. Therefore the product is positive if and only if x is between c and 0.