1)[16] Suppose $p(x) = x \cdot (5 - 3\,x)$ for all real values of x.

a)[6] State a formal definition (involving an appropriate limit) for $p'(2)$.

Solution: $p'(2) \;=\; \lim_{h\to 0} \dfrac{p(2+h) - p(2)}{h}$

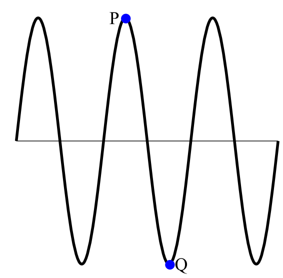
b)[10] Show work that finds the exact value of $p'(2)$ by evaluating the limit you wrote for part (a).

Solution: $p(2) = 2 \cdot (5 - 3 \cdot 2) = -2$ and $p(2+h) = (2+h) \cdot (-1 - 3\,h) = -2 - 7\,h - 3\,h^2$.

$$\begin{aligned} p'(2) \;=\; \lim_{h\to 0} \frac{p(2+h) - p(2)}{h} &= \lim_{h\to 0} \frac{\left(-2 - 7\,h - 3\,h^2\right) - (-2)}{h} \\ &= \lim_{h\to 0} \frac{-7\,h - 3\,h^2}{h} \\ &= \lim_{h\to 0} \frac{h \cdot (-7 - 3\,h)}{h} \\ &= \lim_{h\to 0} (-7 - 3\,h) \\ &= -7 \end{aligned}$$

2)[10] Let f be the sine function. Suppose points P and Q are consecutive highest and lowest points on the graph of f.
[The figure has minimal detail, but may help your analysis.]

Compute the average-rate-of-change for function $f = \sin$ on the interval from P to Q.



Solution: Horizontal spacing between these points is one-half the period of sine. If point P has coordinates $(a, 1)$, then Q has coordinates $(a + \pi, -1)$.

Average-rate-of-change for f is $\dfrac{\sin(a+\pi) - \sin(a)}{(a+\pi) - a} = \dfrac{(-1) - (1)}{\pi} = \dfrac{-2}{\pi}$.

3)[10] Find a value for A so that $f(x) = \begin{cases} A\,\cos(x) & \text{if } x < 0 \\ e^x - A & \text{if } x \geq 0 \end{cases}$ is continuous.

Solution: For any value of A, $A\cos(x)$ and $e^x - A$ are continuous everywhere. This piecewise function will be continuous at 0 if and only if both one-sided limits at 0 have the same value and that limit value is also the function value.

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} A\,\cos(x) = A\,\cos(0) = A$ while $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (e^x - A) = e^0 - A = 1 - A = f(0)$.

Equation $A = 1 - A$ has unique solution $A = \dfrac{1}{2}$.

4)[15] Classify each of the following statements as **TRUE** or **FALSE**, then discuss each classification.
If a statement is false, then either explain why it is false or revise it to be correct; if a statement is true, then provide supporting evidence. Note: 60% of the credit on each part of this problem is for an adequate discussion.

a)[5] If a car's speed is 40 miles-per-hour at 2pm and 60 miles-per-hour at 3pm, then the distance traveled by the car between 2pm and 3pm must be between 40 and 60 miles.

Solution: **FALSE**: No info is given about speeds between 2pm and 3pm. If the car stops at 2:05pm and starts again only at 2:55pm, the distance could be less than 10 miles. Alternatively, car races at 150 mph during 2:05 to 2:55.

b)[5] The derivative of a function f at $x = a$ is the tangent line to the graph of f at $x = a$.

Solution: **FALSE**: The derivative of f at a is slope of the line tangent to graph of f at point $\big(a, f(a)\big)$, it is not that tangent line itself.

c)[5] Suppose $P(t)$ is the quantity [in kilograms] of a chemical produced after t minutes. Also suppose $Q(t)$ is the quantity [in grams] of that chemical produced after t seconds. That implies $P'(t) = 60 \cdot Q'(t)$.

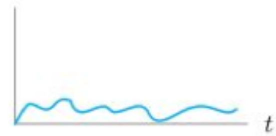
Solution: **FALSE**: Functions P and Q expect their inputs to be in different units; using them together will require a suitable conversion of units for time. If t is measured in minutes, then $P(t)$ and $Q(60\,t)$ compute quantity at the same moment; if time is measured in seconds, then $P(t/60)$ and $Q(t)$ compute quantity at the same moment. Converting output units, between grams and kilograms, complicates the relationship — in any event, the asserted relation is FALSE.

Looking ahead, section 3.4 will show how a specified relation between P and Q implies a consequent relation between their derivative functions.

5)[12] A vehicle moving along a straight road has distance $g(t)$ from its starting point at time t. Each of the following figures is the graph of $g'(t)$ for some context. (Assume scales on the vertical axes are all the same.)

P) 

Q) 

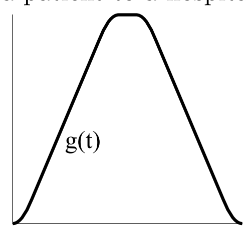
R) 

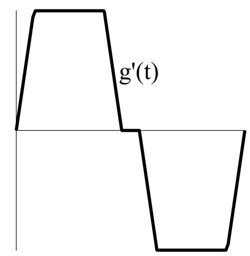
Match each graph with one of the following scenarios (2 points for each match), then sketch plausible graphs of both g and g' for the remaining scenario (6 points for this pair of sketches).

a) A bus drives on a popular route with no traffic.
Solution: figure Q: velocity is zero when bus stops for passengers to enter or leave, the distance between consecutive bus-stops is constant (and so is the time to travel from one to the next).

b) A taxi drives in heavy traffic conditions.
Solution: figure R: speed is low and varies erratically due to traffic

c) An ambulance responds to a medical emergency and then transports a patient to a hospital.
Solution: Ambulance starts at hospital, rushes to location of the medical emergency, is stopped while patient is put into ambulance, then ambulance hurries back to the hospital.



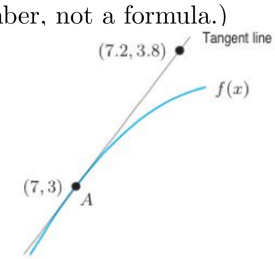


d) A car drives with no traffic and all green lights.
Solution: figure P: car accelerates to speed limit, then cruises at that speed

6)[10] Suppose $F(x) = \dfrac{3\,x + 7x^2}{4x^3 - 5\,x}$. Compute $\lim\limits_{x\to\infty} F(x)$ or explain why that limit does not exist.

Solution: $\lim\limits_{x\to\infty} \dfrac{3\,x+7x^2}{4x^3-5\,x} = \lim\limits_{x\to\infty} \dfrac{(3+7\,x)\cdot x}{(4\,x^2-5)\cdot x} = \lim\limits_{x\to\infty} \dfrac{3+7\,x}{4\,x^2-5} = \lim\limits_{x\to\infty} \dfrac{3+7\,x}{4\,x^2-5}\cdot\dfrac{1/x^2}{1/x^2} = \dfrac{\lim\limits_{x\to\infty} \dfrac{3}{x^2}+\dfrac{7}{x}}{\lim\limits_{x\to\infty} 4-\dfrac{5}{x^2}} = \dfrac{0+0}{4-0} = 0$

7)[12] The figure shows graph of function f, the line tangent to graph of f at point $A = (7, 3)$, and point $(7.2, 3.8)$ on that tangent line. Use this information to complete the following statements. (Each answer is a number, not a formula.)



a)[2] $f(7) = 3$ because $(7, 3)$ is the only known point on graph of f.

b)[6] $f'(7) = \dfrac{3.8-3}{7.2-7} = \dfrac{0.8}{0.2} = 4$ is slope of the tangent line.

c)[4] $f(7.05) \approx f(7) + f'(7)\cdot(7.05-7) = 3 + 4\cdot(0.05) = 3 + 0.2 = 3.2$
This computation uses the tangent line as an approximation to the graph of f in a neighborhood of the point of tangency.

8)[15] In May 2007, the www.census.gov website summarized population changes by reporting there was one birth in the US every eight seconds, one death in the US every thirteen seconds, and one new international immigrant into the US every twenty-seven seconds.

a)[10] Let $f(t)$ be the population of the US, where t is time in seconds measured from the start of May 2007. Find $f'(0)$. Identify the units for your answer.
Solution: $f'(0) = \dfrac{1 \text{ birth}}{8 \text{ sec}} - \dfrac{1 \text{ death}}{13 \text{ sec}} + \dfrac{1 \text{ immigrant}}{27 \text{ sec}} = \dfrac{1}{8} - \dfrac{1}{13} + \dfrac{1}{27}\ \dfrac{\text{person}}{\text{second}}$

b)[5] To the nearest second, how long did it take for the US population to add one person in May 2007?
Solution: $f'(0) = \dfrac{1}{8} - \dfrac{1}{13} + \dfrac{1}{27} = \dfrac{239}{2808}\ \dfrac{\text{person}}{\text{second}}$ and $\dfrac{1}{f'(0)} = \dfrac{2808}{239} \approx 11.75\ \dfrac{\text{seconds}}{\text{person}}$; rounded answer is 12 seconds.

Note: Answering part (b) of problem 8 benefits from access to a calculator, but no other item in this set of problems **requires** use of such a tool.
