Note: “**iff**” is a contraction for “if and only if”.

1. [20] Let $f(x) = x \cdot (x-1)^3$. Its first and second derivatives are

$$f'(x) = (4\,x-1)\cdot(x-1)^2 \qquad \text{and} \qquad f''(x) = 6\cdot(2\,x-1)\cdot(x-1)$$

For each part, give a **brief** explanation. Write NONE for any item where that is appropriate.

a)[2] Find all critical points of f. ____1/4, 1____

Solution: f' is defined everywhere; the critical points of f are the zeros of f', i.e., $\frac{1}{4}$ and 1.

b)[6] Locate where f is increasing and where it is decreasing.

- f is increasing on ____$\left(\frac{1}{4},1\right)$ and $(1,\infty)$____

  Solution: $f'(x) > 0$ iff $x \neq 1$ and $x > \frac{1}{4}$, therefore f increases on $\left(\frac{1}{4},1\right)$ and on $(1,\infty)$.

  [A more complete analysis shows f increases throughout $[\frac{1}{4},\infty)$.]
- f is decreasing on ____$\left(-\infty,\frac{1}{4}\right)$____

  Solution: $f'(x) < 0$ iff $x < \frac{1}{4}$, therefore f is decreasing on $\left(-\infty,\frac{1}{4}\right)$.

c)[6] Locate where, if anywhere, f attains a local extreme.

- f has a local minimum at point(s) with x-coordinate(s)

  Solution: 1/4 is a critical point of f where f' changes from negative to positive, hence f has a local minimum at 1/4. Note: f' does not change sign at 1, the other critical point of f.
- f has a local maximum at point(s) with x-coordinate(s)

  Solution: f has no local maximum; indeed $\lim_{x\to\pm\infty} f(x) = \infty$.

d)[6] Investigate concavity of the graph of f.

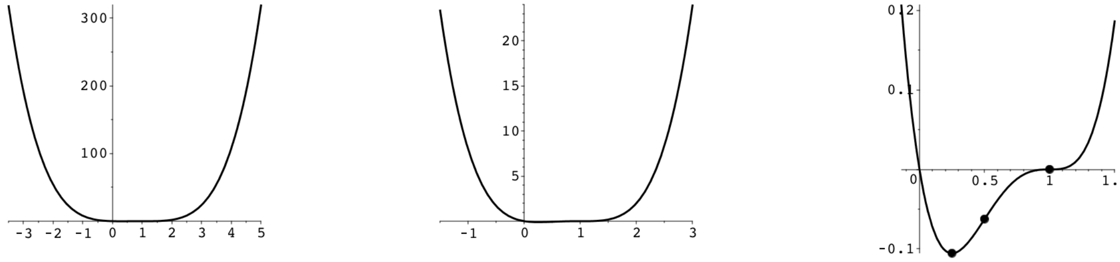
- Graph of f is concave down on ____$(\frac{1}{2},1)$____

  Solution: $f''(x) < 0$ iff $2x-1$ and $x-1$ have opposite signs; this occurs only for interval $\left(\frac{1}{2},1\right)$, that is the only interval where graph of f is concave down.
- Graph of f is concave up on ____$(-\infty,\frac{1}{2})$ and $(1,\infty)$____

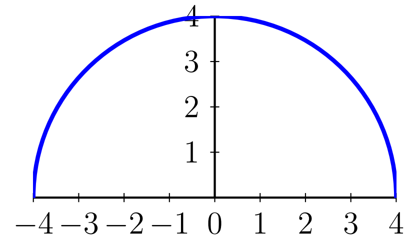
  Solution: $f''(x) > 0$ iff $2x-1$ and $x-1$ have the same sign; that occurs for intervals $\left(-\infty,\frac{1}{2}\right)$ and $(1,\infty)$, that is where graph of f is concave up.
- Graph of f has inflection point(s) with x-coordinate(s) ____1/2, 1____

  Solution: f'' is continuous; $f''(x)$ changes sign at 1/2 and at 1, therefore graph of f has an inflection point at both places.

Here are three graphs of f. After using calculus to analyze f, the third figure uses an x-interval designed to show some crucial features which are invisible in the first two.



2. [15] Upper-half of the curve with equation $x^2 + y^2 = 16$ is shown. Many rectangles fit inside that curve. Consider just those rectangles with one side on the x-axis and the opposite side with its endpoints on the curve. Demonstrate use of calculus ideas to find dimensions of such a rectangle with maximal area.



Solution: Each rectangle being considered has its lower-right corner on the positive x-axis. Let that point be $(x, 0)$; then $\left(x, \sqrt{16 - x^2}\right)$ is upper-right corner, $\left(-x, \sqrt{16 - x^2}\right)$ is upper-left corner, and $(-x, 0)$ is lower-left corner. Area of this rectangle is $g(x) = (2\,x) \cdot \sqrt{16 - x^2}$ with domain $[0, 4]$ which has derivative $g\,'(x) = 2 \cdot \sqrt{16 - x^2} + (2\,x) \cdot \dfrac{-2\,x}{2\sqrt{16 - x^2}} = \dfrac{4\,(8 - x^2)}{\sqrt{16 - x^2}}$. Sign of $g\,'(x)$ is the same as sign of $8 - x^2$, therefore g increases on the interval $\left[0, \sqrt{8}\right)$, reaches its global max at $x = \sqrt{8}$, and decreases on the interval $\left(\sqrt{8}, 4\right]$. The rectangle with maximal area has horizontal side with length $2\sqrt{8}$ and vertical side with length $g\left(\sqrt{8}\right) = \sqrt{8}$.

3. [15] Suppose a point moves in the plane on the parametric curve

$$x(t) = 1 + \cos(t) \qquad \text{and} \qquad y(t) = t + \sin(2\,t) \qquad \text{with} \qquad -0.5 \le t \le 3.5$$

a)[3] Show this particle never comes to a stop.

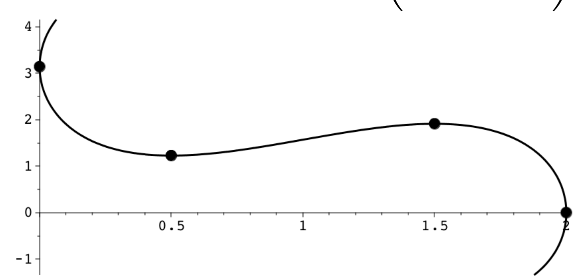
Solution: $x'(t) = -\sin(t)$ and $y'(t) = 1 + 2\cos(2\,t)$. $x'(t) = 0$ iff t is a multiple of π; for those values of t, we find $y'(n\,\pi) = 1 + 2\cos(2\,n\,\pi) = 1 + 2 = 3$. Hence there is no choice of t such that $x'(t) = 0 = y'(t)$.

b)[6] For $-0.5 \le t \le 3.5$, is the particle ever moving straight up or straight down? If so, identify the t-value(s), the point(s) $\big(x(t), y(t)\big)$, and identify the direction (up or down).

Solution: Solution for part (a) has relevant information. For t in $[-0.5, 3.5]$, $x'(t) = 0$ for $t = 0$ and $t = \pi$ with $y'(t) = 3$ for both. The curve is going straight up at $\big(x(0), y(0)\big) = (2, 0)$ and at $\big(x(\pi), y(\pi)\big) = (0, \pi)$

c)[6] For $-0.5 \le t \le 3.5$, is the particle ever moving straight horizontally, left or right? If so, identify the t-value(s), the point(s) $\big(x(t), y(t)\big)$, and identify the direction (right or left).

Solution: $y'(t) = 0$ iff $\cos(2\,t) = \dfrac{-1}{2}$ which has solutions $\dfrac{\pi}{3}$ and $\dfrac{2\pi}{3}$ in $[-0.5, 3.5]$. Since $x(t)$ decreases on $[0, \pi]$, the particle is moving straight to the left at $\left(x\left(\frac{\pi}{3}\right), y\left(\frac{\pi}{3}\right)\right) = \left(\dfrac{3}{2}, \dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}\right)$ and at $\left(x\left(\dfrac{2\,\pi}{3}\right), y\left(\dfrac{2\,\pi}{3}\right)\right) = \left(\dfrac{1}{2}, \dfrac{2\,\pi}{3} - \dfrac{\sqrt{3}}{2}\right)$.



4. [10] Show work which computes the following limits.

a) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x}}{\ln(x)} =$

Solution: This limit has the indeterminate form $\dfrac{\infty}{\infty}$. Consider limit for ratio of derivatives.

$$\lim_{x\to\infty} \frac{\frac{d}{dx}\sqrt{x}}{\frac{d}{dx}\ln(x)} = \lim_{x\to\infty} \frac{1/\left(2\sqrt{x}\right)}{1/x} = \lim_{x\to\infty} \frac{x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{2} = \infty.$$

Because that limit exists, l'Hôpital's Rule implies $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x}}{\ln(x)} = \lim_{x\to\infty} \frac{\frac{d}{dx}\sqrt{x}}{\frac{d}{dx}\ln(x)} = \infty.$

b) $\displaystyle\lim_{x\to 0} \frac{e^{-x} - 1 + x}{x^2} =$

Solution: This limit has the indeterminate form $\dfrac{0}{0}$. The following sequence of calculations are evidence that two uses of l'Hôpital's Rule produces an explicit limit value.

$$\lim_{x\to 0} \frac{e^{-x} - 1 + x}{x^2} = \lim_{x\to 0} \frac{\frac{d}{dx}\left(e^{-x} - 1 + x\right)}{\frac{d}{dx}x^2} = \lim_{x\to 0} \frac{-e^{-x} + 1}{2\,x} = \lim_{x\to 0} \frac{\frac{d}{dx}\left(-e^{-x} + 1\right)}{\frac{d}{dx}(2x)} = \lim_{x\to 0} \frac{e^{-x}}{2} = \frac{1}{2}.$$

5. [15] An object moves along a straight line. At time $t = 0$, its velocity is $2\,\dfrac{\text{ft}}{\text{s}}$. For the next fifteen seconds, the object accelerates at the rate of $3\,\dfrac{\text{ft/s}}{\text{s}}$.

a)[4] Write a formula for velocity as a function of time t (for $0 \le t \le 15$).

Solution: Since $v'(t) = a = 3$ is constant, it is immediate that $v(t) = a\,t + v(0) = 3\,t + 2$ feet-per-second.

b)[4] Write a definite integral which computes the distance traveled during the first six seconds.

Solution: $\displaystyle\int_0^3 v(t)\;dt = \int_0^3 3\,t + 2\;dt$

c)[3] Will a left Riemann sum with $\Delta t = 1$ seconds for your answer to part (b) yield an under-estimate, exact value, or an over-estimate? Explain. (Note: this part does not ask you to do the actual left sum calculation.)

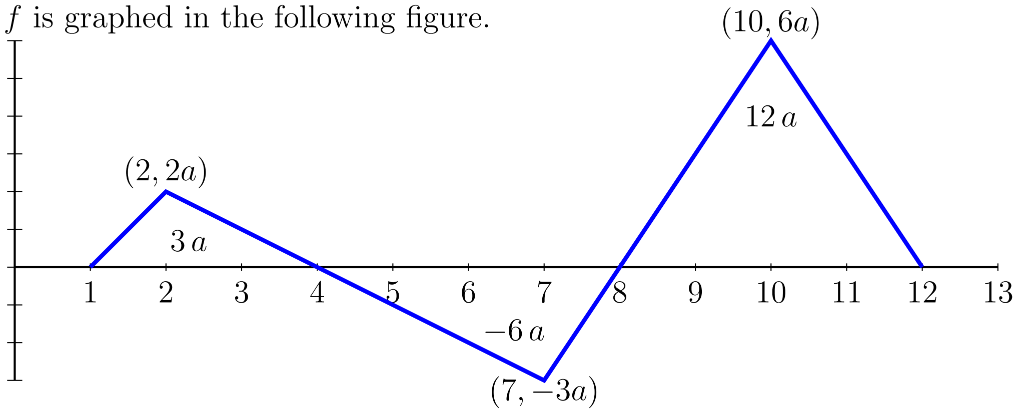
Solution: Because the integrand (the velocity function) is increasing, a left sum will yield an under-estimate for exact value of the definite integral.

d)[4] Use the Fundamental Theorem of Calculus to compute exact value for your part (b) answer.

Solution: $3\,\dfrac{t^2}{2} + 2\,t$ is an antiderivative for the velocity function, thus

$$\int_0^3 3\,t + 2\;dt = \left(3\,\frac{t^2}{2} + 2\,t\right)\Bigg|_0^3 = \left(3\,\frac{9}{2} + 2\cdot 3\right) - \left(3\,\frac{0}{2} + 2\cdot 0\right) = 19.5 \text{ feet}$$

6.[10] Function f is graphed in the following figure.



Consider the following definite integrals — arrange them in order from smallest to largest. Your answer can cite just the upper-case letters, e.g., respond R,T,Q,S if $R < T < Q < S$.

$$A = \int_1^4 f(x)\,dx \qquad B = \int_1^8 f(x)\,dx \qquad C = \int_1^{12} f(x)\,dx \qquad D = \int_5^5 f(x)\,dx$$

$$E = \int_4^8 f(x)\,dx \qquad F = \int_4^{12} f(x)\,dx \qquad G = \int_8^{12} f(x)\,dx$$

Solution: Most of the answer "E , B , D , A , F , C , G" can be found using the easy observation that the triangle underneath the x-axis has area larger than the left triangle and smaller than the right triangle. The comparison of A and F is the only one needing some extra attention. A more careful analysis could define a with the statement $f(2) = 2a$, that implies $f(7) = -3a$ and $f(10) = 6a$ [remember graph of f consists of straight line segments]. Therefore

$$E = -6a \ < \ B = -3a \ < \ D = 0 \ < \ A = 3a \ < \ F = 6a \ < \ C = 9a \ < \ G = 12a$$

7. [15] Classify each of the following statements as **TRUE** or **FALSE**, then discuss each classification.

If a statement is false, then either explain why it is false or revise it to be correct; if a statement is true, then provide supporting evidence. Note: 60% of the credit on each part of this problem is for an adequate discussion (**but** an irrelevant statement such as "2 + 1 = 3" does not qualify as a correct revision for any calculus statement).

a) If $f'(c) = 0$, then $f(x)$ has either a local minimum or a local maximum at $x = c$.
Solution: **FALSE** $f(x) = (x-1)^3$ has $f'(x) = 3\,(x-1)^2$ with $f'(1) = 0$, but $f(1) = 0$ is neither a local min nor local max for f [which is strictly increasing throughout $(-\infty, \infty)$].

b) Every cubic polynomial has an inflection point.
Solution: **TRUE** Let $p(x) = a\,x^3 + b\,x^2 + c\,x + d$ where a, b, c, d are constants **and** $a \neq 0$ [to guarantee p is a cubic]. Linear polynomial $p''(x) = 6\,a\,x + 2\,b$ changes sign at $k = \dfrac{-2\,b}{6\,a} = \dfrac{-b}{3\,a}$, so graph of cubic $p(x)$ has an inflection point at $x = \dfrac{-b}{3\,a}$.

c) Suppose f and g are continuous on $[2, 5]$. If f is increasing but g is decreasing on $[2, 5]$, then

$$\int_2^5 f(x)\,dx \ \neq \ \int_2^5 g(x)\,dx$$

Solution: **FALSE** For f and g graphed below, the two definite integrals have the same value.