1. [10] Let $f(t)$ be the depth (in centimeters) of water in a tank at time t (in minutes).

   a) [2] What does the sign of $f'(t)$ tell us?
   Solution: Water depth is increasing during time-intervals when f' has positive value, depth is decreasing when that derivative has negative values.

   b) [4] Explain the meaning of $\boldsymbol{f'(30) = 20}$ [include units]
   Solution: Water depth is increasing $20 \dfrac{\text{cm}}{\text{min}}$ when $t = 30$ minutes.

   c) [4] Use the information in part (b), at time $t = 30$ minutes, to find the rate-of-change for depth in meters with respect to time in hours at time t in hours.
   Solution: At time $t = 30$ min $= 0.5$ hours, the **rate of change for water depth** was
   $$20 \frac{\text{cm}}{\text{min}} = \left(20 \frac{\text{cm}}{\text{min}}\right) \cdot \left(\frac{60 \text{ min}}{\text{hr}}\right) \cdot \left(\frac{1 \text{ m}}{100 \text{ cm}}\right) = 12 \ \frac{\text{meters}}{\text{hour}}$$

2. [10] Let $g(x) = \ln(-3\,x)$.

   a) [2] What is the (maximal) domain of g?
   Solution: Input to a logarithm must be positive: $-3\,x > 0$ is equivalent to $x < 0$.

   b) [2] What is the range of g?
   Solution: Both ln and g have range $(-\infty, \infty)$.

   c) [6] Write an expression for $g'(x)$ [simplify your answer].
   Solution: $g'(x) = \ln'(-3\,x) \cdot \dfrac{d}{dx}(-3\,x) = \dfrac{1}{-3\,x} \cdot (-3) = \dfrac{1}{x}$

3. [15] Let $f(x) = 3 + 2\,e^{-0.5\,x}$

   a) [7] Find the *local linearization* of f at $x = 0$.
   (This is also called the *Tangent Line Approximation* for f near $x = 0$.)
   Solution: $f'(x) = 0 + 2\,e^{-0.5\,x} \cdot (-0.5) = -\,e^{-0.5\,x}$. The tangent line approximation for f at 0 is
   $$L(x) \;=\; f(0) + f'(0) \cdot (x - 0) \;=\; (3 + 2) + (-1) \cdot x \;=\; 5 - x$$

   b) [5] Use your answer for part (a) to compute an approximation of $f(0.5)$.
   Solution: $f(0.5) \;\approx\; L(0.5) \;=\; 5 - 0.5 \;=\; 4.5$

   c) [3] Identify a feature of the graph of f (a sketch might be useful) that lets you decide whether your answer to part (b) is an over-estimate or an under-estimate for the true value of $f(0.5)$.
   Solution: $f''(x) = 2\,e^{-0.5\,x} \cdot (-0.5)^2$ is positive everywhere, hence graph of f is concave up on every interval and that graph lies **above** each of its tangent lines. That implies $L(0.5)$ is an **under-estimate** for $f(0.5)$.

4. [12] The Bay of Fundy (Atlantic coast of Canada) is known for extreme tides. Depth (in meters) of water in the Bay of Fundy is modeled as a function of time t (hours after midnight) by the function

$$f(t) = 10 + 7.5\cos\left(\frac{\pi}{6}t\right)$$

a) [4] Find derivative function(s) suitable to answer the items in part (b).

Solution: $f'(t) = 0 + 7.5\left(-\sin\left(\frac{\pi}{6}t\right)\right)\cdot\frac{\pi}{6} = \frac{-5\pi}{4}\cdot\sin\left(\frac{\pi}{6}t\right)$

b) [8] How quickly is the tide rising or falling (in meters-per-hour) at the following times?

- 6:00 am **Sol**: $t = 6$ and $f'(6) = (-5\pi/4)\sin(\pi) = 0\,\frac{\text{m}}{\text{hr}}$ (depth is at its minimum)
- 9:00 am **Sol**: $f'(9) = (-5\pi/4)\sin(1.5\pi) = (-5\pi/4)\cdot(-1) \approx 3.93\,\frac{\text{m}}{\text{hr}}$ (rising tide)
- noon **Sol**: $t = 12$ and $f'(12) = (-5\pi/4)\sin(2\pi) = 0\,\frac{\text{m}}{\text{hr}}$ (depth is at its maximum)
- 6:00 pm **Sol**: $t = 18$ and $f'(18) = (-5\pi/4)\sin(3\pi) = 0\,\frac{\text{m}}{\text{hr}}$ (depth is back to its min)

Note: Example 6 on page 152 uses 0.507 instead of $\pi/6$ to get a more realistic model for the water depth. That model for the tidal oscillation has period $(2\pi)/0.507 \approx 12.4$ hours while this problem's model has period $(2\pi)/(\pi/6) = 12$ hours.

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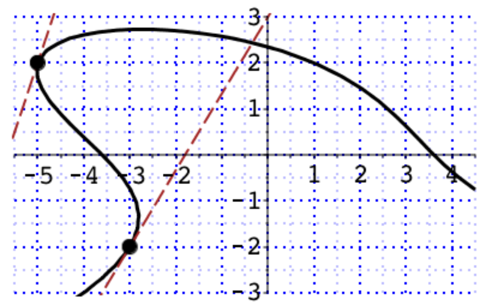
5. [18] The figure shows points (x, y) satisfying equation

$$x^2 + 2xy + y^3 = 13$$



a) [8] Write an expression for $\frac{dy}{dx}$.

Hint: use implicit differentiation.

Solution: Differentiation with-respect-to x, isolation of the y' terms, etc., yields

$$2x + 2\,(1\cdot y + x\cdot y') + 3y^2\cdot y' = 0$$

$$\left(2x + 3y^2\right)\cdot y' = 2\,x\cdot y' + 3y^2\cdot y' = -2x - 2y = -2\,(x+y)$$

$$\frac{dy}{dx} = y' = \frac{-2\,(x+y)}{2x + 3y^2}$$

b) [4] Compute slope of the tangent at point $(-3, -2)$.

Solution: $\left.\frac{dy}{dx}\right|_{(-3,-2)} = \frac{-2\cdot\left((-3)+(-2)\right)}{2\cdot(-3) + 3\cdot(-2)^2} = \frac{-2\cdot(-5)}{-6+12} = \frac{10}{6} = \frac{5}{3}$

c) [6] Write an equation for the tangent at point $(-5, 2)$. [Note: not the point in part (b).]

Solution: $\left.\frac{dy}{dx}\right|_{(-5,2)} = \frac{-2\cdot\left((-5)+2\right)}{2\cdot(-5) + 3\cdot 2^2} = \frac{-2\cdot(-3)}{-10+12} = \frac{6}{2} = 3$. Tangent equation: $y = 2 + 3\cdot(x+5)$

6. [10] This problem asks you to use differentiation facts to find some <u>*antiderivatives*</u>.

a) Find a function f such that $f'(x) = e^{0.5x} + x$

Solution: $\frac{d}{dx}\left(a\,e^{0.5x} + b\,x^2\right) = a\,e^{0.5x}\cdot 0.5 + b\cdot(2x)$. Solve $a\cdot(0.5) = 1$ and $b\cdot 2 = 1$ to get $a = \frac{1}{0.5} = 2$ and $b = \frac{1}{2}$ which yields answer $f(x) = 2\,e^{0.5x} + \frac{1}{2}x^2$.

b) Find a function g such that $g''(x) = \sin(3\,x)$

Solution: $\dfrac{d^2}{dx^2}\Big(a\cdot\sin(3\,x)\Big) = \dfrac{d}{dx}\Big(a\cdot\cos(3\,x)\cdot 3\Big) = a\cdot\Big(-\sin(3\,x)\Big)\cdot 3^2 = -9\cdot a\cdot\sin(3\,x).$

Let $a = \dfrac{1}{-9}$ to choose $g(x) = \dfrac{-1}{9}\sin(3\,x)$.

7. [15] Classify each of the following statements as **TRUE** or **FALSE**, then discuss each classification.

If a statement is false, then either explain why it is false or revise it to be correct; if a statement is true, then provide supporting evidence. Note: 60% of the credit on each part of this problem is for an adequate discussion (**but** an <u>irrelevant</u> statement such as "2 + 1 = 3" does not qualify as a correct revision for any calculus statement).

a) If $g''(x) > 0$ for all x, then g' is a decreasing function.

Solution: **FALSE**: $f' > 0$ on interval (a, b) implies f increases on (a, b); apply here with $f = g'$

b) If f and g are two functions whose first and second derivatives exist, then

$$\big(f\cdot g\big)'' = (f''\cdot g) + (f\cdot g'')$$

Solution: **FALSE**: apply Product Rule **twice**:

$$\big(f\cdot g\big)'' = (f'\cdot g + f\cdot g')' = (f''\cdot g) + 2\,(f'\cdot g') + (f\cdot g'')$$

c) If $g(x) = f(-2\,x)$ and $f'(x) > 0$ for all x, then g is a decreasing function.

Solution: **TRUE**: $g'(x) = f'(-2\,x)\cdot(-2) = (\textbf{positive})\cdot(-2) < 0$ implies g is decreasing.

8. [10] Definition: **Relative-Rate-of-Change** for differentiable function g is $\mathbf{R}(g) = \dfrac{g'}{g}$

a) Compute Relative-Rate-of-Change for $g(x) = e^{-3\,x}$. Simplify your answer.

Solution: $\mathbf{R}(g) = \mathbf{R}\left(e^{-3\,x}\right) = \dfrac{e^{-3\,x}\cdot(-3)}{e^{-3\,x}} = -3$

Note: $\mathbf{R}\left(a\,e^{k\,x}\right) = k$ and $\mathbf{R}\left(a\,b^{x}\right) = \ln(b)$; indeed, $\dfrac{d}{dx}\ln\Big(g(x)\Big) = \dfrac{g'(x)}{g(x)}$

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b) Show Relative-Rate-of-Change has the following property:

$$\mathbf{R}(f\cdot g) = \mathbf{R}(f) + \mathbf{R}(g)$$

Hint: Use the Product Rule, simplify stuff, then interpret the simplified result.

Solution: $\mathbf{R}(f\cdot g) = \dfrac{(f\cdot g)'}{f\cdot g} = \dfrac{f'\cdot g + f\cdot g'}{f\cdot g} = \dfrac{f'\cdot g}{f\cdot g} + \dfrac{f\cdot g'}{f\cdot g} = \dfrac{f'}{f} + \dfrac{g'}{g} = \mathbf{R}(f) + \mathbf{R}(g)$

Note: If f is increasing by 5% and g is decreasing by 2%, then $f\cdot g$ is increasing by $5\% + (-2\%) = 3\%$.

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9. [15] Suppose g is differentiable such that $1 \le g'(x) \le 2$ for all x; also suppose $g(0) = 4$.

a) [3] Explain why the information given above implies $g(3)$ can not be equal to 4.
Solution: $g'(x) \ge 1 > 0$ implies g is an increasing function, that implies $g(3) \ne g(0) = 4$

b) [4] Explain why the information given above implies $g(3)$ can not be equal to 20.
Solution: If $g(3)$ did equal 20, then average rate of change for g on interval $[0, 3]$ would be

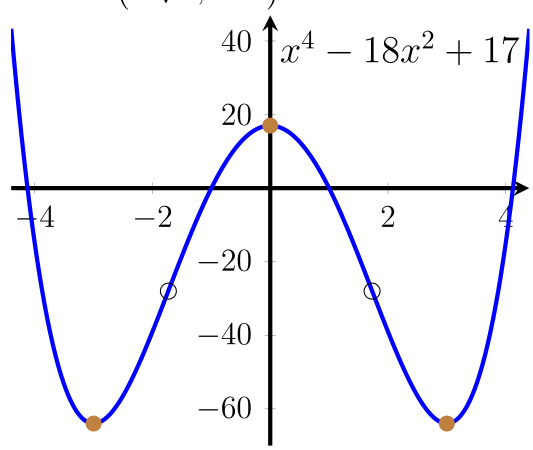
$$\frac{g(3) - g(0)}{3 - 0} = \frac{20 - 4}{3 - 0} = \frac{16}{3} \approx 5.33$$

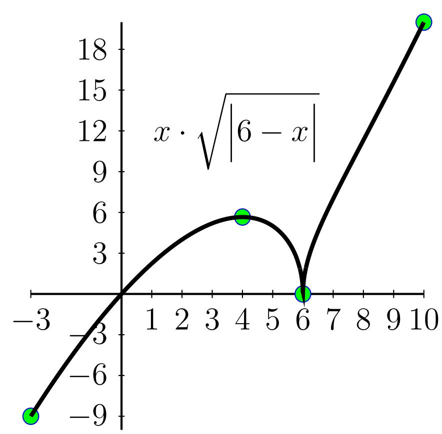
The Mean Value Theorem says there is some c in interval $[0, 3]$ where $g'(c)$ has the same value, but the bounds on g' values say that is impossible!

c) [8] Show work which finds **Best Possible Bounds** for the value of $g(3)$.
Solution: If $g'(x) = 1$ throughout $[0, 3]$, then $g(3)$ would be $g(0) + 1 \cdot (3 - 0) = 4 + 3 = 7$; on the other hand, if $g'(x) = 2$ throughout $[0, 3]$, then $g(3)$ would be $g(0) + 2 \cdot (3 - 0) = 4 + 6 = 10$. In conclusion, $7 \le g(3) \le 10$ presents the **Best Possible Bounds** for $g(3)$.

10. [10] Show work which locates all critical points and inflection points of $f(x) = x^4 - 18\,x^2 + 17$.

Solution: $f'(x) = 4\,x^3 - 18 \cdot 2\,x = 4 \cdot x \cdot (x^2 - 9) = 4 \cdot x \cdot (x - 3) \cdot (x + 3)$; the critical points are 0, 3, -3 where f' has value zero. $f''(x) = 4 \cdot 3x^2 - 36 = 12 \cdot (x^2 - 3) = 12 \cdot \left(x + \sqrt{3}\right) \cdot \left(x - \sqrt{3}\right)$; f'' changes from positive to negative at $-\sqrt{3}$ and back to positive at $\sqrt{3}$, each of those locates an inflection point for f. More explicitly, f has its global minimum at $(\pm 3, -64)$ and a local maximum at $(0, 17)$, its inflection points are $\left(\pm\sqrt{3}, -28\right)$.





11. [15] Locate all extremes, local and global, of $p(x) = x \cdot \sqrt{|6 - x|}$ on closed interval $[-3, 10]$.

Note: this function can also be written in the form $p(x) = \begin{cases} x \cdot \sqrt{6 - x} & \text{if} \quad x \le 6 \\ x \cdot \sqrt{x - 6} & \text{if} \quad x > 6 \end{cases}$

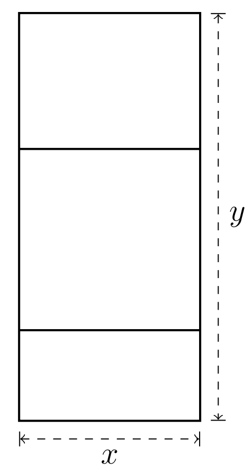
Solution: $p'(x) = \begin{cases} 1 \cdot \sqrt{6 - x} \; + \; x \cdot \dfrac{1}{2\sqrt{6 - x}} \cdot (-1) \\ 1 \cdot \sqrt{x - 6} \; + \; x \cdot \dfrac{1}{2\sqrt{x - 6}} \cdot 1 \end{cases} = \begin{cases} \dfrac{3}{2} \cdot \dfrac{4 - x}{\sqrt{6 - x}} & \text{if} \quad x < 6 \\ \dfrac{3}{2} \cdot \dfrac{x - 4}{\sqrt{x - 6}} & \text{if} \quad x > 6 \end{cases}$

- $p'(4) = 0$ and $p(4) = 4\sqrt{2}$ is a local maximum [sign of p' changes from positive to negative there]
- $p'(6)$ is not defined; sign of p' changes from negative to positive at $x = 6$ so $p(6) = 0$ is local min
- $p(-3) = -3 \cdot \sqrt{6 - (-3)} = -3 \cdot \sqrt{9} = -3 \cdot 3 = -9$ is global minimum value
- $p(10) = 10 \cdot \sqrt{|6 - 10|} = 10 \cdot \sqrt{4} = 10 \cdot 2 = 20$ is global maximum value

12. [10] Find area of the largest rectangle with one side on the x-axis and two upper corners on the graph of $y = 27 - x^2$.

Solution: Put lower-right corner of the rectangle on x axis at $(t, 0)$ where $0 \le t \le \sqrt{27}$; the other corners are at $(\pm t, 6 - t^2)$ and $(-t, 0)$. Area of this rectangle is $A = g(t) = (2\,t) \cdot (27 - t^2) = 54\,t - 2\,t^3$. Therefore $\dfrac{dA}{dt} = g'(t) = 54 - 2 \cdot \left(3\,t^2\right) = 6 \cdot \left(9 - t^2\right) = 6 \cdot (3 - t) \cdot (3 + t)$. The unique critical point for this function is $t = 3$ where that derivative changes from positive to negative. The rectangle with maximal area has corners at $(\pm 3, 0)$ and $(\pm 3, 18)$; its area is $\Big(3 - (-3)\Big) \cdot \Big(27 - 3^2\Big) = 6 \cdot 18 = 108$.

13. [20] Three adjacent rectangular garden plots will share some fencing. This figure shows the desired configuration (with two internal fences, each shared by a pair of gardens).



a) Suppose 100 feet of fencing is available for use on this project. What is the maximal area (total area of all three gardens) which can be enclosed?

Solution: Let y be length of a vertical side in this figure and x be length of a horizontal side. The total amount of fencing is $2\,y + 4\,x = 100$ and the area is $A = x \cdot y$. The fencing constraint implies $y = 50 - 2\,x$ and $A = x \cdot (50 - 2\,x) = 50\,x - 2\,x^2$ [with domain $0 \le x \le (100/4) = 25$]. Therefore $\dfrac{dA}{dx} = 50 - 4\,x$ and the critical point for the area function is $x = 50/4 = 12.5$ feet. The largest total area is $12.5 \times 25 = 312.5$ square-feet. **Note**: half of the fencing is used for the two long vertical fences and the other half is used for the four shorter horizontal fences.

b) Suppose the three gardeners agree that they will be happy if total area of their gardens is 400 square-feet. What is the minimal length of fencing that will be needed?

Solution: Area constraint $x \cdot y = 400$ implies $y = \dfrac{400}{x}$ and the total fencing is $P = 4\,x + 2\,y = 4\,x + 2 \cdot \dfrac{400}{x}$ [with domain $x > 0$]. Therefore $\dfrac{dP}{dx} = 4 - \dfrac{800}{x^2}$ and the unique positive critical point for the perimeter function is

$$x = \sqrt{\frac{800}{4}} = \sqrt{200} = 10\,\sqrt{2} \approx 14.14 \text{ feet}$$

The other dimension will be $y = \dfrac{400}{10\,\sqrt{2}} = \dfrac{40}{\sqrt{2}} = 20\,\sqrt{2} \approx 28.28$ feet; total fencing is $4 \times \left(10\,\sqrt{2}\right) + 2 \times \left(20\,\sqrt{2}\right) = 80 \times \sqrt{2} \approx 113.14$ feet.