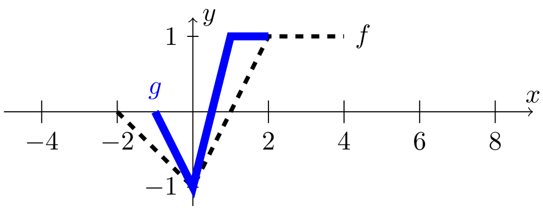
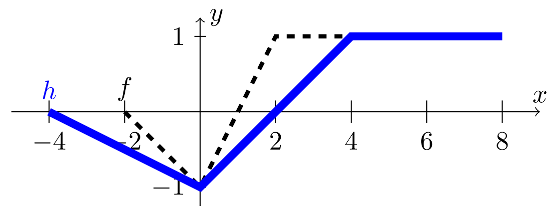
1. The graph of $y = f(x)$ is shown 4 times. Sketch graphs of g, h, G and H in parts (a)–(d).

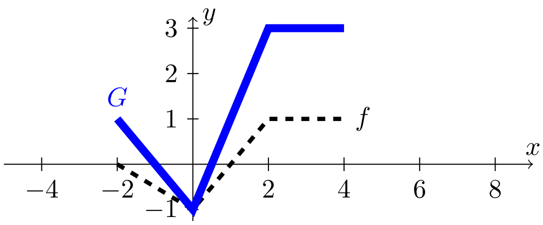
a) $g(x) = f(2\,x)$ [compress horizontally]



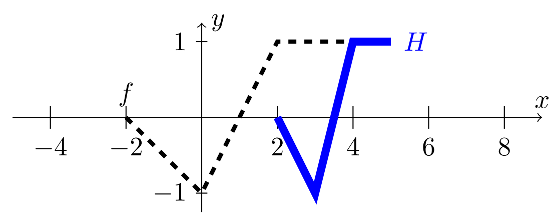
b) $h(x) = f\left(\dfrac{x}{2}\right)$ [stretch horizontally]



c) $G(x) = 2\,f(x) + 1$ [stretch vertically, shift up]



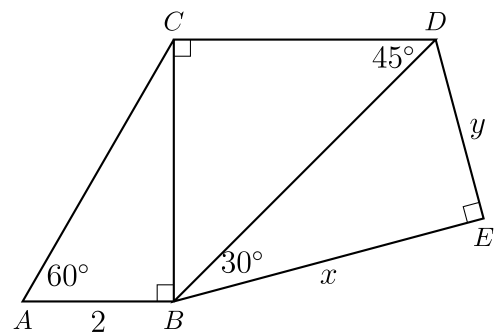
d) $H(x) = f(2\,x - 6) = g(x - 3)$ [shift g right by 3]



e) Compute $g(0.5) = f(1) = 0$, $g(2) = f(4) = 1$, $h(-2) = f(-1) = -1/2$, $h(6) = f(3) = 1$

Compute $G(-2) = 2 \cdot 0 + 1 = 1$, $G(2) = 2 \cdot 1 + 1 = 3$, $H(2) = f(-2) = 0$, $H(5) = f(4) = 1$

2. Find the <u>*exact*</u> values of lengths x and y in this figure.



Solution: Vertical segment BC has length $2 \cdot \tan(60°) = 2 \cdot \sqrt{3} = \sqrt{12}$. Right triangle BCD is isosceles because angle BDC is 45°, thus horizontal segment CD also has length $\sqrt{12}$ and the Pythagorean Theorem implies hypotenuse BD has length

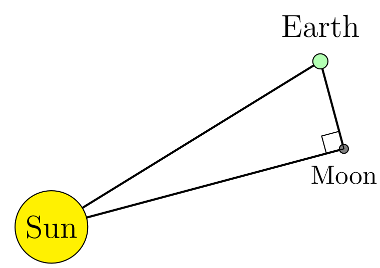
$$\sqrt{(BC)^2 + (CD)^2} = \sqrt{12 + 12} = \sqrt{24}$$

Segment BD is also the hypotenuse of right triangle BED. Side y is opposite the 30° angle, therefore $y = (BD) \cdot \sin(30°) = \sqrt{24} \cdot \dfrac{1}{2} = \sqrt{6}$. Furthermore, side x is adjacent to the 30° angle implies

$$x = (BD) \cdot \cos(30°) = \sqrt{24} \cdot \frac{\sqrt{3}}{2} = \sqrt{18} = 3 \cdot \sqrt{2}$$

3. When the moon is exactly **half-full**

- Earth, Moon and Sun form a right angle (see the figure [but it's not to-scale]),
- the angle formed by Sun, Earth and Moon is measured to be 89.85°.



If the Earth–Moon distance is 240,000 miles, estimate the distance between Earth and Sun.

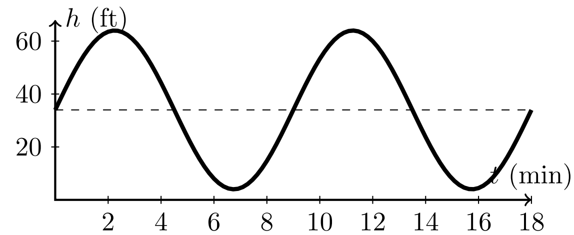
Solution: $\cos(89.85°) = \dfrac{\text{Earth:Moon}}{\text{Earth:Sun}} = \dfrac{240\,000}{\text{Earth:Sun}}$ implies

$$\text{Earth:Sun} = \frac{240\,000}{\cos(89.85°)} \approx 91\,673\,352 \approx 91.6 \text{ million miles}$$

4. A Ferris wheel completes a turn every 9 minutes, has radius 30 feet, and its boarding platform is 4 feet above the ground.

a) Find the height above ground for a person at the 2 o'clock position.
Solution: The 2 o'clock position is one-third the way around (in counter-clockwise direction) from the 6 o'clock position; more usefully for a computation using the sine function, it is one-twelfth around from the 3 o'clock position.
Height above ground is $4 + 30 + 30 \sin\left(\frac{1}{12} \cdot 360°\right) = 34 + 30 \cdot \sin(30°) = 34 + 30 \cdot \frac{1}{2} = 34 + 15 = 49$ feet.

b) Sketch a graph for the height f(t), in feet, if at t = 0 the person is at the 3 o'clock position, going up.



c) If $g(t) = 3f(t)$, find amplitude and period of function g; interpret in terms of the height and rotation speed of a different ride.
Solution: f has amplitude 30 feet and g has amplitude $3 \cdot 30 = 90$ feet; both f and g have period 9 minutes. Function g gives height on a Ferris wheel with radius 90 feet and 9 minutes per revolution.

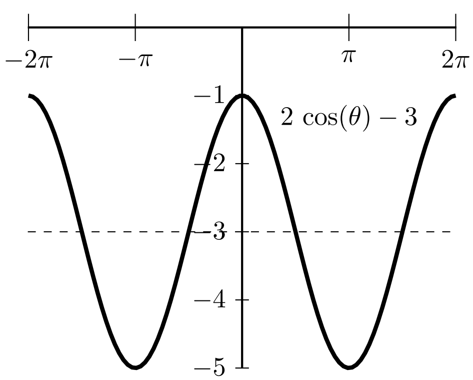
d) If $h(t) = f(3t)$, find amplitude and period of function h; interpret in terms of height and rotation speed of another ride.
Solution: f and h both have amplitude 30 feet; h has period $9/3 = 3$ minutes. Function h gives height on the same Ferris wheel when it goes 3 times faster.

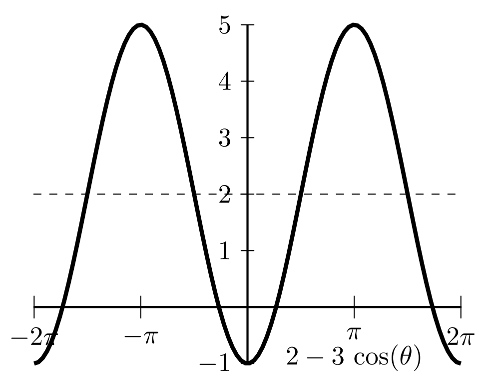
5. Let θ be the acute angle (an angle between 0° and 90°) such that $\sin(\theta) = 1/5$.

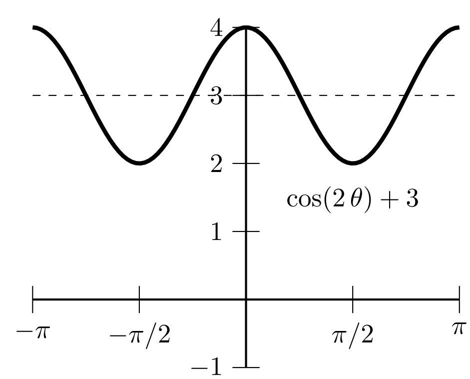
a) $\cos(\theta) = \sqrt{1 - \sin(\theta)^2} = \sqrt{1 - (1/5)^2} = \sqrt{24}/5$
b) $\sin(180° - \theta) = \sin(\theta) = 1/5$
c) $\cos(\theta - 90°) = \cos(90° - \theta) = \sin(\theta) = 1/5$

6. For each of the functions below, find its period, midline, amplitude and sketch its graph.

a) $f(x) = 2\cos(\theta) - 3$: period is 2π, midline has equation $y = -3$, amplitude is 2
b) $g(x) = 2 - 3\cos(\theta)$: period is 2π, midline has equation $y = 2$, amplitude is $|-3| = 3$
c) $h(x) = \cos(2\theta) + 3$: period is $(2\pi)/2 = \pi$, midline has equation $y = 3$, amplitude is 1







7. The height in inches of the tip of the minute hand on a vertical clock face is a function, $h(t)$, of the time t in minutes. The minute hand is 4 inches long, and the middle of the clock face is 90 inches above the ground.

   a) Find the midline, amplitude and period of this function.
   Solution: Midline is $h = 90$ inches, amplitude is 4 inches, and period is 60 minutes.

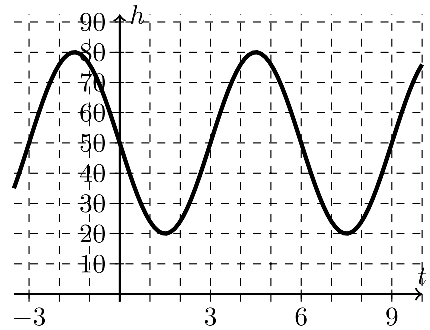
   b) How high is the tip of the minute hand at 12:40 pm?
   Solution: At 40 minutes past some hour, the minute hand points at the clock's 8 o'clock position; the counter-clockwise angle from the 3 o'clock position is $180 + 30 = 210°$. Height of minute hand's tip is

$$90 + 4 \cdot \sin(210°) = 90 - 4 \cdot \sin(30°) = 90 - 4 \cdot \frac{1}{2} = 88 \text{ inches}$$

   c) Give a formula for h(t) if t = 0 at noon. **Check**: $h(40)$ computes the answer to part (b).
   Solution: If we use radians as units for inputs to sine and cosine, then 60 minutes as period for the minute hand implies our function will have structure $A\,\mathrm{trig}\left(\frac{2\pi}{60}(t-h)\right) + k = A\,\mathrm{trig}\left(\frac{\pi}{30}t + \phi\right) + k$. If the time-of-day is t minutes after noon, then angle between 3 o'clock position and the minute hand is $\frac{\pi}{2} - \frac{\pi}{30}t$. Therefore

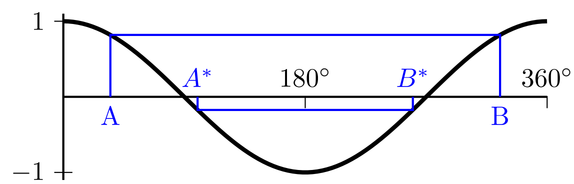
$$h(t) = 90 + 4 \cdot \sin\left(\frac{\pi}{2} - \frac{\pi}{30}t\right) = 90 + 4 \cdot \cos\left(\frac{\pi}{30}t\right)$$

8. Find amplitude, midline and period for this graph.
Amplitude is $\frac{80-20}{2} = 30$, midline has equation $h = \frac{80+20}{2} = 50$, period is 6.



   a) Give a formula for $h = f(t)$ choosing a trigonometric function which does not require horizontal shifts.
   Solution: $h = 50 - 30\sin\left(\frac{2\pi}{6}t\right)$

   b) Give another possible formula for $h = f(t)$.
   Solution: Shift a cosine curve with the same period to the left by 1.5 (i.e., a quarter-period). $h = 50 + 30\cos\left(\frac{\pi}{3}(t + 1.5)\right) = 50 + 30\cos\left(\frac{\pi}{3}t + \frac{\pi}{2}\right)$



9. The righthand figure above has a cosine graph; A and B are related by $\cos(A) = \cos(B)$.

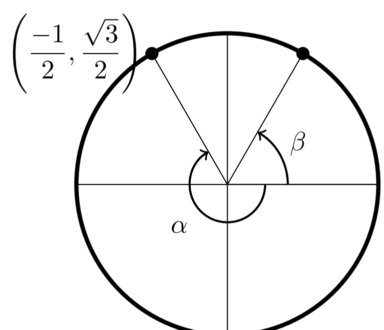
   a) If $A = 35°$, find B.
   Solution: $B = 360° - A = 360° - 35° = 325°$

   b) Draw $A^* = 100°$ on the figure, then find B^* (value in degrees) with same cosine.
   Solution: $B^* = 360° - A^* = 360° - 100° = 260°$

10. Transform radians into degrees and vice-versa (give both an exact answer and an answer accurate to two decimal places):

   a) $5.3 \text{ rad} = 5.3 \times \frac{180}{\pi} = \frac{954}{\pi} \approx 303.667\,6314 \approx 303.67°$

   b) $12° = 12 \times \frac{\pi}{180} = \frac{\pi}{15}$ radians $\approx 0.209\,439\,5102 \approx 0.21$ radians

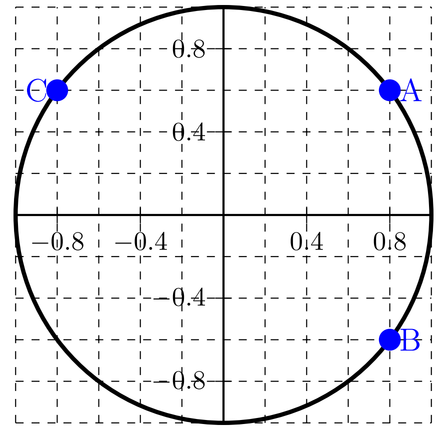


11. Let $\alpha = \frac{-4\pi}{3}$ (radians). Report the exact values (not decimal approximations) of $\sin(\alpha)$, $\cos(\alpha)$, $\tan(\alpha)$, $\cot(\alpha)$, $\sec(\alpha)$, and $\csc(\alpha)$.
Solution: Reference angle for α is $\beta = \frac{\pi}{3}$. Because each angle in an equilateral triangle has size β, we know $\cos(\beta) = \frac{1}{2}$ and $\sin(\beta) = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$ as exact values. Therefore

$$\sin(\alpha) = \sin(\beta) = \frac{\sqrt{3}}{2}, \qquad \cos(\alpha) = -\cos(\beta) = \frac{-1}{2}, \qquad \tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3},$$

$$\csc(\alpha) = \frac{1}{\sin(\alpha)} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}, \qquad \sec(\alpha) = \frac{1}{\cos(\alpha)} = \frac{1}{-1/2} = -2, \qquad \cot(\alpha) = \frac{\cos(\alpha)}{\sin(\alpha)} = \frac{-1/2}{\sqrt{3}/2} = \frac{-1}{\sqrt{3}}$$

12. Angle θ determines a point on The Unit Circle with coordinates $(0.8, 0.6)$. Mark points A, B, C corresponding to angles θ, $-\theta$, $\pi - \theta$ respectively. Then find the <u>exact</u> value for each of:



a) $\cos(\theta) = 0.8, \quad \sin(\theta) = 0.6, \quad \tan(\theta) = \dfrac{0.6}{0.8} = \dfrac{3}{4} = 0.75$

b) $\cos(-\theta) = 0.8, \quad \sin(-\theta) = -0.6, \quad \tan(-\theta) = \dfrac{-0.6}{0.8} = \dfrac{-3}{4} = -0.75$

c) $\cos(\pi-\theta) = -0.8, \quad \sin(\pi-\theta) = 0.6, \quad \tan(\pi-\theta) = \dfrac{0.6}{-0.8} = \dfrac{3}{-4} = -0.75$

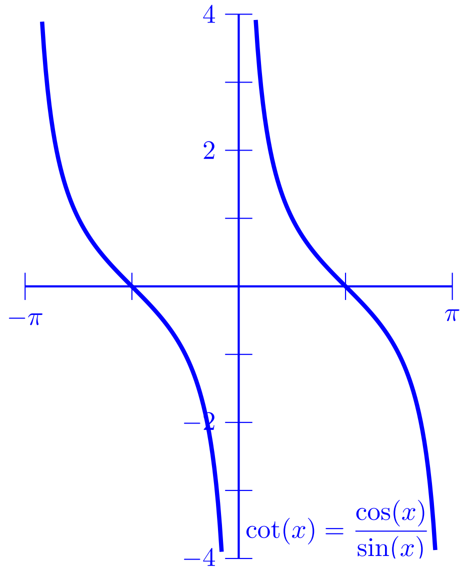
d) $\sec(\theta) = \dfrac{1}{\cos(\theta)} = \dfrac{1}{4/5} = \dfrac{5}{4} = 1.25, \quad \csc(\theta) = \dfrac{1}{\sin(\theta)} = \dfrac{1}{3/5} = \dfrac{5}{3},$

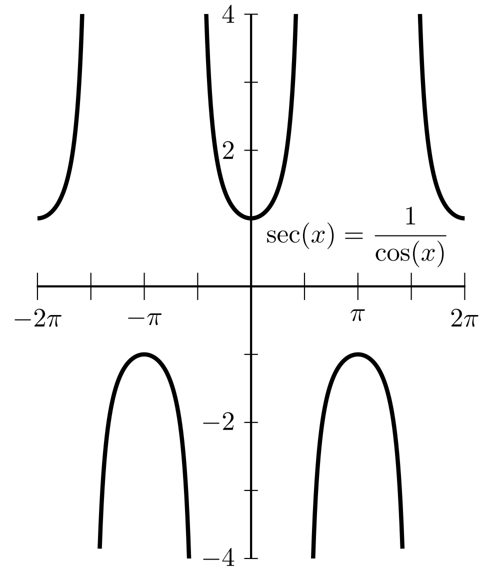
$\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)} = \dfrac{0.8}{0.6} = \dfrac{4}{3}$

13. At what values of t does the graph of $f(t) = \tan(\pi t)$ have vertical asymptotes? Those values are the same as the zeros of $\cos(\pi t)$. Is that a coincidence? Explain.

Solution: The tangent function has vertical asymptotes at odd multiples of $\dfrac{\pi}{2}$; that implies function f has vertical asymptotes at odd multiples of $\dfrac{1}{2}$. In both cases, $\tan(t) = \dfrac{\sin(t)}{\cos(t)}$ and $f(t) = \tan(\pi t) = \dfrac{\sin(\pi t)}{\cos(\pi t)}$, location of the vertical asymptote is center of a short interval where value of the numerator function stays close to 1 or -1 while the denominator is zero at that center and remains close to zero elsewhere in the interval.

14. Sketch graphs of COTANGENT: $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$ and SECANT: $\sec(\theta) = \dfrac{1}{\cos(\theta)}$. For each function (cot and sec): identify its domain and range, identify its period, give equation(s) for any asymptote(s) of its graph, and locate the x- and y-intercepts (if any).





domain: omit multiples of π for domain of cot; omit odd multiples of $\pi/2$ for domain of sec

vertical asymptote: each of cot and sec has a vertical asymptote at every real number which is not in its domain

y-intercept: $\sec(0) = 1$; cot does not have a y-intercept because 0 is not in its domain

range: range of cot is the set of all real numbers; range of sec is $(-\infty, -1] \cup [1, \infty)$.

x-intercept: $\cot(x) = 0$ if and only if x is an odd multiple of $\pi/2$; sec does not have any x-intercept

period: sec has period 2π while the period of cot is π