This collection of problems involves ideas we have studied in chapters 4 and 5 plus section 6.1.
An actual test will have fewer problems and will provide space for work to be shown.

1. [5] The amount (in mg) of a drug in the body decreases exponentially at the rate of 24% per hour; the initial amount was 60 mg. Write a formula for the amount of this drug in the body after t hours.

   Solution: Hourly rate is $r = -0.24$ and hourly factor is $b = 1 + r = 0.76$. After t hours, there are $60 \cdot \left(0.76^t\right)$ mg remaining in the body.

2. [5] A population is growing at the rate of 1.5% a year. By what percent does it grow in 8 years?

   Solution: Annual rate is $r = 0.015$ and annual factor is $b = 1 + r = 1.015$. Therefore, the 8-year factor is $1.015^8 \approx 1.126\,493$; subtracting 1 from that and rescaling to percentage terms, the 8-year growth rate is about $12.65\,\%$.

3. [5] Value of some old machinery has decreased exponentially. If the 6–year decrease was 20%, by what percent did it decrease in just one of those years?

   Solution: If the six-year rate was $r_6 = -0.20$, then the six-year factor was $b_6 = 1+r_6 = 0.80$. If b is the one-year factor, then $b^6 = b_6 = 0.80$ implies $b = 0.80^{1/6} \approx 0.963\,492$ and the one-year rate is $r = b - 1 \approx -0.036\,508 \approx -3.65\,\%$. (I.e., annual decrease was about 3.65 percent.)

4. [12] Let $P(t)$ be the population (millions of people) in a country at time t (years after 1990).
   Use the following information (data) for both parts of this problem: $P(4) = 2.75$ and $P(12) = 3.19$.

   a) Find a formula for $P(t)$ assuming it is linear. Describe, in words, changes in the country's population given this assumption (linear model).

      Solution: Average rate of change was $\dfrac{3.19 - 2.75}{12 - 4} = \dfrac{0.44}{8} = 0.055$. Therefore

      $$P(t) \;=\; 2.75 \;+\; 0.055\,(t-4) \;=\; 3.19 \;+\; 0.055\,(t-12) \;=\; 2.53 \;+\; 0.055\,t$$

      In 1990, the population was 2.53 million and increasing by $55,000$ people–per–year.

   b) Find a formula for $P(t)$ assuming it is exponential. Describe, in words, changes in the country's population given this assumption (exponential model).

      Solution: Suppose $P(t) = a \cdot b^t$. Then $b^8 = b^{12-4} = \dfrac{a \cdot b^{12}}{a \cdot b^4} = \dfrac{P(12)}{P(4)} = \dfrac{3.19}{2.75} = 1.16$. This computation is sufficient to report $P(t) = (2.75) \cdot \left(1.16^{(t-4)/8}\right)$ with the interpretation that the population was 2.75 million in $1990 + 4 = 1994$ and was growing by $16\,\%$ every eight years.

      More computation produces extra detail. $b = 1.16^{1/8} \approx 1.018\,726$ and $a = \dfrac{2.75}{b^4} = \dfrac{2.75}{\sqrt{1.16}} \approx 2.553$ yield $P(t) = (2.553) \cdot \left(1.018\,726^t\right)$ which means the population was 2.553 million in 1990 and was growing by $1.87\,\%$ per year.

5. [10] Suppose \$4700 is deposited in an account paying a nominal interest rate of 1.8% per year.

   a) If the interest is compounded semiannually (twice a year), then how much is in the account after 5 years?

      Solution: $\$4700 \left(1 + \dfrac{0.018}{2}\right)^{(2)\cdot(5)} \approx \$4700 \cdot (1.093\,733\,873) \approx \$5140.549\,203 \approx \$5140.55$

   b) If interest is compounded semiannually, then the effective annual rate is ____1.808____ % per year. (Round your numerical answer to 3 decimal places, but show your work below.)

      Solution: The annual factor is $b = \left(1 + \dfrac{0.018}{2}\right)^2 = 1.009^2 = 1.018\,081$, so the effective annual rate is $r = b - 1 = 0.018\,081 = 1.8081\,\% \approx 1.808\,\%$

6. [12] In the tables below, assume function f is linear and function g is exponential. Complete each table for $x = 0$ and $x = 4$, then compute $f(7)$ and $g(7)$. If an answer is not an integer, write it as a fraction or as a decimal with 3-decimal place accuracy.

linear

x	$y = f(x)$
0	$4 - 2 \cdot m = 4 - 6 = -2$
2	4
4	$4 + 2 \cdot m = 4 + 6 = 10$
6	16
7	$16 + m = 16 + 3 = 19$

exponential

x	$y = g(x)$
0	$4/b^2 = 4/2 = 2$
2	4
4	$4 \cdot b^2 = 4 \cdot 2 = 8$
6	16
7	$16 \cdot b = 16 \cdot \sqrt{2} \approx 22.627$

Solution: Slope of f is $m = \dfrac{16-4}{6-2} = \dfrac{12}{4} = 3$ and $f(x) = f(2) + m \cdot (x-2) = 4 + 3 \cdot (x-2) = 3\,x - 2$. If $g(x) = a \cdot b^x$, then $b^4 = b^{6-2} = \dfrac{a \cdot b^6}{a \cdot b^2} = \dfrac{g(6)}{g(2)} = \dfrac{16}{4} = 4$ implies

$$b = \left(b^4\right)^{1/4} = 4^{1/4} = \left(2^2\right)^{1/4} = 2^{2/4} = 2^{1/2} = \sqrt{2}$$

therefore $a = g(2)/b^2 = 4/2 = 2$. The result of that work is $g(x) = 2 \cdot \left(\sqrt{2}\right)^x$.

Note: $f(4) = \dfrac{f(2) + f(6)}{2}$ and $g(4) = \sqrt{g(2) \cdot g(6)}$.

Caution: While a report of intermediate computations might show a rounded value, subsequent calculations **should use the full precision** obtained by your calculator. That task can be aided by regular use of a calculator's **store** button. For the exponential function, rounding $b = \sqrt{2}$ to 1.414 might produce an inaccurate value for a, compute $g(4) \approx 7.995$ [accurate to only two decimal places], and estimate $g(7) \approx 22.604$ [accurate to only one decimal place]. Such errors would be minimized or avoided by using the stored value $b = 4^{1/4} \approx 1.414\,213\,562\,3731$

7. [12] A radioactive substance decays at a continuous rate of 16% per year, and 150 mg of the substance is present in the year 2000.

a) Write a formula for the amount present, A (in mg), t years after 2000.

Solution: Because the problem specified a **continuous rate** of $-16\% = -0.16$ per year, the solution will use a base-e exponential function: $A = f(t) = f(0)\,e^{-0.16\,t} = 150\,e^{-0.16\,t}$ mg.

b) Estimate when the quantity drops below 10 mg.

Solution: $f(t) = 10$ is equivalent to $e^{-0.16\,t} = \dfrac{10}{150} = \dfrac{1}{15}$ and $-0.16\,t = \ln\left(e^{-0.16\,t}\right) = \ln\left(\dfrac{1}{15}\right)$. The solution is $t = \dfrac{\ln(1/15)}{-0.16} \approx 16.925\,314$ years (after 2000) which corresponds to early December in 2016.

Caution: this change is continuous, solving $f(t) = 9$ or $f(t) = 9.99$ is not equivalent.

8. [16] The amount (mg) of a drug in the body t hours after taking a pill is given by $A(t) = 21\left(0.82^t\right)$.

a) What percent of the drug is lost each hour?

Solution: Hourly factor $b = 0.82$ implies hourly rate is $r = b - 1 = 0.82 - 1 = -0.18$ which means 18 % is lost each hour.

b) What is the half-life of this drug?

Solution: $\dfrac{1}{2} = 0.82^t$ is equivalent to $\log\left(\dfrac{1}{2}\right) = \log\left(0.82^t\right) = t\,\log(0.82)$ with solution $t = \dfrac{\log(1/2)}{\log(0.82)} \approx 3.492\,788\,621 \approx 3.49$ hours

c) Rewrite the equation for this drug as $A(t) = a\, e^{k\, t}$
Solution: $b = e^k$ implies $k = \ln(0.82) \approx -0.198\,450\,939$ and

$$A(t) = 21 \left(e^k\right)^t = 21 \left(e^{\ln(0.82)}\right)^t \approx 21\, e^{-0.19845\, t}$$

d) What is the continuous decay rate per hour of this drug?
Solution: $k = \ln(0.82) \approx -0.19845 = -19.845\,\%$ implies the continuous decay rate is $19.845\,\%$ per hour.

9. [14] Solve for t (give *both* exact solution and a three-decimal place approximation).

a) $4 \cdot \ln(t-5) = 6$
Solution: The problem's equation is equivalent to the following

$$\begin{aligned} \ln(t-5) &= \frac{6}{4} = \frac{3}{2} \\ t - 5 &= e^{3/2} \\ t &= 5 + e^{3/2} \approx 9.481\,689\,070 \approx 9.482 \end{aligned}$$

b) $10 \cdot \left(1.04^t\right) = 8 \cdot e^{0.05\, t}$
Solution: Applying base-e logarithm to right-hand-side and left-hand-side of that equation yields

$$\begin{aligned} \ln(\text{RHS}) &= \ln\left(8 \cdot e^{0.05\, t}\right) = \ln(8) + \ln\left(e^{0.05\, t}\right) = \ln(8) + 0.05 \cdot t \\ \ln(\text{LHS}) &= \ln\Big(10 \cdot \left(1.04^t\right)\Big) = \ln(10) + \ln\left(1.04^t\right) = \ln(10) + t \cdot \ln(1.04) \end{aligned}$$

Therefore $RHS = LHS$ is equivalent to $\ln(RHS) = \ln(LHS)$ and $0.05 \cdot t - t \cdot \ln(1.04) = \ln(10) - \ln(8)$ with solution $t = \dfrac{\ln(10) - \ln(8)}{0.05 - \ln(1.04)} \approx 20.701\,142\,338 \approx 20.701$

10. [12] Sound in decibels is measured by comparing its intensity [energy per area] I to a benchmark sound intensity I_0 [benchmark is barely audible, its intensity is 10^{-16} watts/cm^2]. Then,

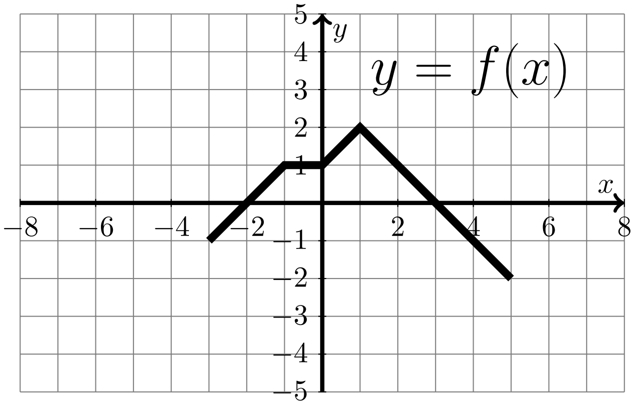
$$\text{Noise level in decibels} \;=\; 10 \,\log\left(\frac{I}{I_0}\right)$$

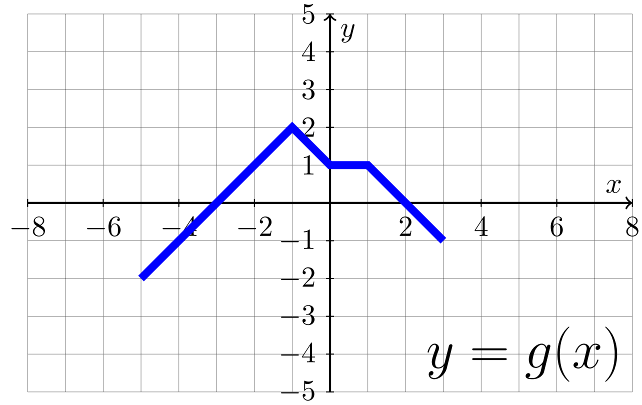
Suppose sound A measures 30 decibels. Also suppose intensity of sound B is 5 times larger than the intensity of sound A. Show work that finds the decibel rating of sound B [round to nearest integer].

Solution: Let I_A and I_B be the intensities for sounds A and B; the problem states $I_B = 5 \times I_A$. The task is to find the corresponding relation between $dB_A = 30$ decibels and dB_B, the decibel measures for those sounds.

$$\begin{aligned} dB_B &= 10 \,\log\left(\frac{I_B}{I_0}\right) \\ &= 10 \,\log\left(\frac{5 \times I_A}{I_0}\right) = 10 \,\log\left(5 \times \frac{I_A}{I_0}\right) \\ &= 10 \,\left[\log(5) + \log\left(\frac{I_A}{I_0}\right)\right] \\ &= 10 \,\log(5) \;+\; 10 \,\log\left(\frac{I_A}{I_0}\right) \\ &= 10 \,\log(5) \;+\; dB_A \\ &= 10 \,\log(5) \;+\; 30 \\ &\approx 6.989700043 + 30 \approx 36.9897 \\ &\approx 37 \,\text{decibels} \end{aligned}$$

11. [14] The graph of function f is given below; the domain of f is $[-3, 5]$.





a) [3] Use the formula $g(x) = f(-x)$ and the graph of f to compute $g(-1) =$

Solution: $g(-1) = f\big(-(-1)\big) = f(1) = 2$

b) [6] Sketch the graph of $g(x) = f(-x)$ in the righthand grid.

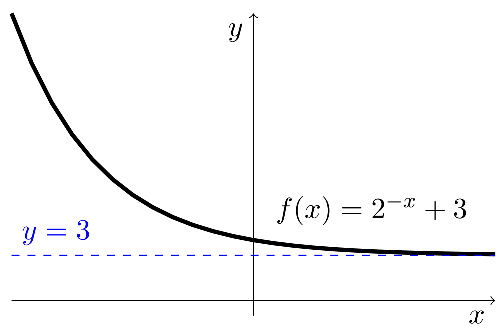
Solution: Reflect the graph of f across the y-axis. Graph of f consists of straight line segments connecting the points $(-3,-1)$, $(-1,1)$, $(0,1)$, $(1,2)$, $(5,-2)$. The graph of $f(-x)$ has line segments connecting points $(-5,-2)$, $(-1,1)$, $(0,1)$, $(1,1)$, $(3,-1)$.

c) [5] Find all x–intercepts for the graph of function $g(x) = f(-x)$.

Solution: The x–intercepts for the graph of f are -2 and 3. That implies $g(2) = f(-2) = 0$ and $h(-3) = f\Big(-(-3)\Big) = f(3) = 0$. Because $g(x) = f(-x)$ has domain $[-5, 3]$, the only solutions of $h(x) = 0$ are 2 and -3.

12. [10] Functions $f(x) = 2^{-x} + 3$ and $g(x) = \ln(x+3)$ are graphed below. Respond to 5 items for each.

a)
- Sketch the asymptote to the graph of $f(x) = 2^{-x} + 3$
- The asymptote to graph of f has equation $\underline{\quad y = 3 \quad}$
- Graph of f has y–intercept equal to $\underline{\quad 2^0 + 3 = 4 \quad}$
- Domain of f is $\underline{\quad (-\infty, \infty) \quad}$
- Range of f is $\underline{\quad (3, \infty) \quad}$



b)
- Sketch the asymptote to the graph of $g(x) = \ln(x+3)$
- The asymptote to graph of g has equation $\underline{\quad x = -3 \quad}$
- Graph of g has x–intercept equal to $\underline{\quad -2 \text{ since } \ln(1) = 0 \quad}$
- Domain of g is $\underline{\quad (-3, \infty) \quad}$
- Range of g is $\underline{\quad (-\infty, \infty) \quad}$