M 151– Spring 2017 – Practice for Test 1 — Solutions

1. (a) Town A: $\dfrac{\text{change in } P}{\text{change in } t} = \dfrac{25-5}{50-0} = \dfrac{20}{50} = \dfrac{2}{5}$

   Town B: $\dfrac{\text{change in } P}{\text{change in } t} = \dfrac{20-10}{50-0} = \dfrac{10}{50} = \dfrac{1}{5}$

   (b) Both towns have positive average rate of change, Town A has the larger rate ($2/5 > 1/5$) and is growing faster.

   (c) Town B starts with more people than Town A: $10 > 5$.

   (d) Population of each town is a linear function of time t (years since 1950):

   $$\text{Town A: } P_A = \frac{2}{5}t + 5; \qquad \text{Town B: } P_B = \frac{1}{5}t + 10;$$

   Populations are equal when $\frac{2}{5}t + 5 = P_A = P_B = \frac{1}{5}t + 10$ which is equivalent to

   $$\frac{1}{5}\,t = \frac{2}{5}\,t - \frac{1}{5}\,t = 10 - 5 = 5$$

   with solution $t = 25$ when $P_A = (2/5) \cdot (25) + 5 = 15$ and $P_B = (1/5) \cdot (25) + 10 = 15$. The populations were the same at year $1950 + 25 = 1975$ when both populations were 15 thousand.

2. (a) Top row of table lists three values of t in increasing order ($1.5 < 2.4 < 3.6$) and bottom row with corresponding values of $R(t)$ is also in increasing order: $-5.7 < -3.1 < -1.4$. As t increases, the value of $R(t)$ also increases: $t_1 < t_2$ implies $R(t_1) < R(t_2)$. Thus R is an increasing function.

   (b) For interval $[1.5, 2.4]$, $\dfrac{\text{change in } R(t)}{\text{change in } t} = \dfrac{-3.1-(-5.7)}{2.4-1.5} = \dfrac{2.6}{0.9} \approx 2.889$

   For interval $[2.4, 3.6]$, $\dfrac{\text{change in } R(t)}{\text{change in } t} = \dfrac{-1.4-(-3.1)}{3.6-2.4} = \dfrac{1.7}{1.2} \approx 1.417$

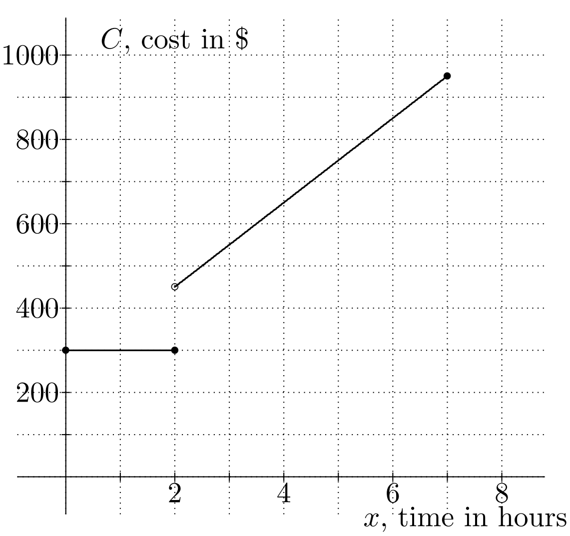
   Rate of change for R is decreasing as we move left to right; a function whose rate-of-change is always decreasing has a graph which is concave down.

3. $C(x) = \begin{cases} 300\ , & 0 \le x \le 2 \\ 300 + 150 + 100(x-2)\ , & 2 < x \le 7 \end{cases}$

   or

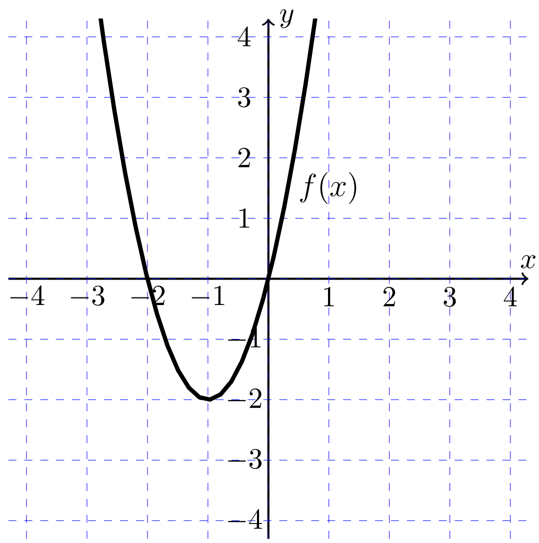
   $C(x) = \begin{cases} 300\ , & 0 \le x \le 2 \\ 250 + 100x\ , & 2 < x \le 7 \end{cases}$

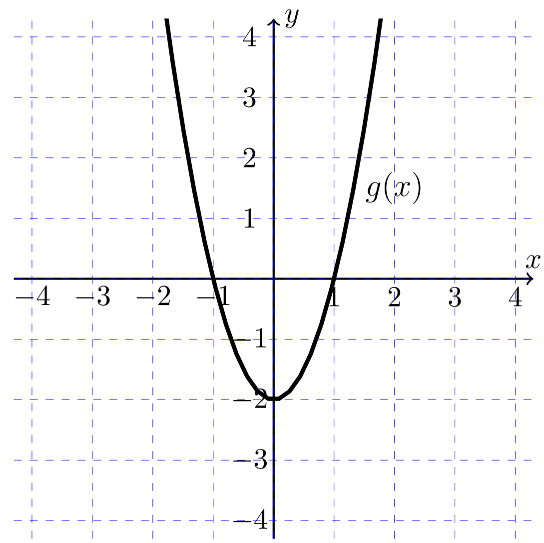
   One person might argue 0 should not be included in the domain (no hours, no cost) while another person could argue that a "no show" would still cost $300.



4. Let $f(x) = 2x^2 + 4x$:

   (a) The graph of $g(x) = f(x-1)$ is the graph of f shifted one unit to the right, i.e., a parabola passing through the points $(-1,0)$,$(0,-2)$,$(1,0)$.





   (b) The following work writes and simplifies a formula for $g(x) = f(x-1)$.

$$\begin{aligned} f(x-1) &= 2\,(x-1)^2 + 4\,(x-1) \\ &= 2\,\left(x^2 - 2\,x + 1\right) + 4\,(x-1) \\ &= \left(2\,x^2 - 4\,x + 2\right) + (4\,x - 4) \\ &= 2\,x^2 - 2 \end{aligned}$$

5. Context: a circle with diameter d has area $f(d)$ and a circle with radius r has area $\pi\, r^2$, adding pepperoni to a pizza with area A costs $g(A)$, and a \$3 package of pepperoni covers 250 square-inches of pizza,

   (a) A circle whose diameter is 16 inches has an $8 = 16/2$ inch radius; area of a 16 inch circle is $f(16) = \pi \left(\frac{16}{2}\right)^2 = 64\,\pi$ square-inches.

   (b) Since \$3 buys pepperoni to cover 250 square-inches of pizza, covering a pizza whose area is A square-inches costs $g(A) = (\$3) \cdot \frac{A}{250} = \frac{\$3}{250} \cdot A = 0.012\,A$ dollars. Adding pepperoni to a 200 square-inch pizza costs $g(200) = (0.012) \times (200) = 2.4 = \$2.40$

   (c) The cost of adding pepperoni to a 12 inch diameter pizza is

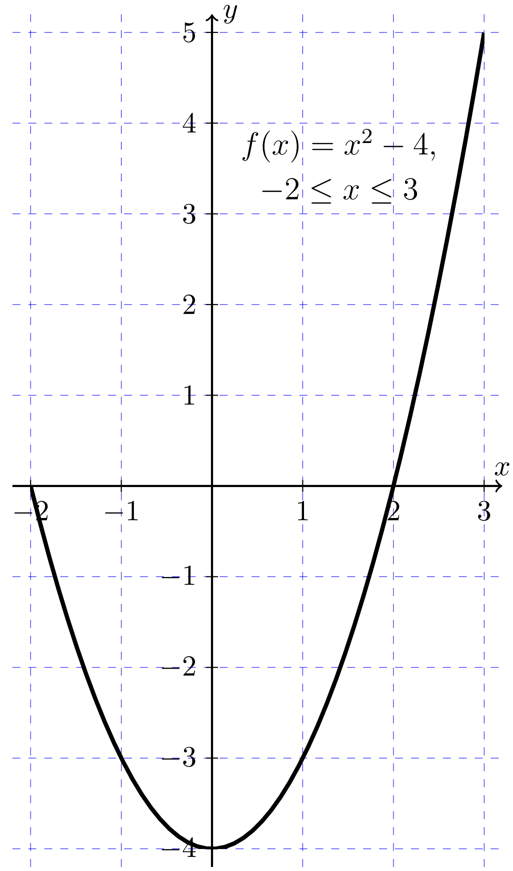
$$g\big(f(12)\big) = g\left(\pi\, 6^2\right) = (0.012) \times \pi \times 36 = (0.432) \times \pi \approx 1.36 \text{ dollars}$$

   (d) $C = g(A)$ is pepperoni cost (dollars) for a pizza with area A (square-inches); therefore $g^{-1}(C) = A$ is pizza area which can be pepperoni'd for C dollars.

   Solving $1.50 = g(A) = 0.012\,A$ yields $A = g^{-1}(1.50) = \frac{1.50}{0.012} = 125$ square-inches.

   Note: since \$3 "covers" 250 square-inches, it should be no surprise that half of that named cost will "cover" half of that named area.

6. $f(x) = x^2 - 4$ is graphed for $-2 \le x \le 3$;
this figure shows a **complete graph** of this function.



Lowest point on this graph is $\big(0, f(0)\big) = (0, -4)$;
highest point on this graph is $\big(3, f(3)\big) = (3, 5)$.

Range of this function *on its domain*,
i.e., range of $x^2 - 4$ on $[-2, 3]$,
is $\big[f(0), f(3)\big] = [-4, 5]$.

Note: Interval $[f(-2), f(3)] = [0, 5]$
is NOT the range for this function.

7. $C = f(q) = 250 + 0.7\,q$ is cost (thousand dollars) to produce q pounds of an alloy.

   (a) $f(10) = 250 + (0.7) \times (10) = 250 + 7 = 257$ thousand dollars is cost to produce 10 pounds of alloy.

   (b) Solve $C = f(q) = 250 + 0.7\,q$ for q in terms of C. The following equations are equivalent.

$$\begin{aligned} 250 + 0.7\,q &= C \\ 0.7\,q &= C - 250 \\ q &= \frac{C - 250}{0.7} = f^{-1}(C) \end{aligned}$$

   (c) $f^{-1}(257) = \dfrac{257 - 250}{0.7} = \dfrac{7}{0.7} = 10$ pounds is the amount of alloy which can be produced at cost of 257 thousand dollars. [Notice the intrinsic relation between parts (a) and (c).]

8. $f^{-1}(V)$ gives the time in *seconds* it takes for an accelerating car to reach a speed of V km/hr.

9. Consider the function $f(x) = 2x^2 - 4x - 30$

   (a) $f(x) = 2\cdot\big(x^2 - 2\,x - 15\big)$ and $x^2 - 2\,x = \big(x^2 - 2\,x\big) + (1 - 1) = \big(x^2 - 2\,x + 1\big) - 1 = (x-1)^2 - 1$ imply $f(x) = 2 \cdot \Big[\big((x-1)^2 - 1\big) - 15\Big] = 2 \cdot \big[(x-1)^2 - 16\big] = 2 \cdot (x-1)^2 - 32.$

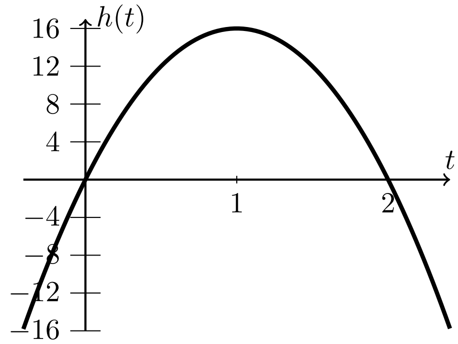
   (b) $f(x) = 2 \cdot \big(x^2 - 2\,x - 15\big) = 2 \cdot \big(x^2 - 2\,x - 3 \cdot 5\big) = 2 \cdot (x+3) \cdot (x-5)$

   (c) Solve $f(x) = 2x^2 - 4x - 30 = 8$. That equation is equivalent to $2 \cdot \big(x^2 - 2x\big) = 8 + 30 = 38$; dividing by 2 yields $x^2 - 2x = 19$. Add 1 to complete the square: $(x-1)^2 = 19 + 1 = 20$. Therefore $x - 1 = \pm\sqrt{20}$ whose solutions are $x_1 = 1 + \sqrt{20}$ and $x_2 = 1 - \sqrt{20}$.

10. Factor, find zeros (horizontal intercepts), and graph.

(a)

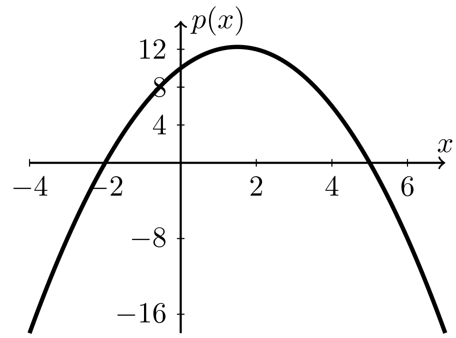
$$h(t) = -16\,t^2 + 32\,t$$
$$= -16 \cdot \left(t^2 - 2\,t\right)$$
$$= -16 \cdot t \cdot (t-2)$$



$h(t) = 0$ is equivalent to $t = 0$ or $t - 2 = 0$; hence the horizontal intercepts for graph of h are at $t = 0$ [the origin] and $t = 2$.

(b)

$$p(x) = -x^2 + 3\,x + 10$$
$$= -1 \cdot \left(x^2 - 3\,x - 10\right)$$
$$= -1 \cdot \left(x^2 - 3\,x - 2 \cdot 5\right)$$
$$= -1 \cdot (x+2) \cdot (x-5)$$
$$= (x+2) \cdot (5-x)$$



$p(x) = 0$ is equivalent to $x + 2 = 0$ or $5 - x = 0$; hence the horizontal intercepts for graph of p are at $x = -2$ and $x = 5$.

11. (a) The following equations are equivalent.

$2\,z^2 - 20\,z + 10 = 0$		
$z^2 - 10\,z + 5 = 0$	because	divide by 2
$z^2 - 10\,z = -5$	because	subtract 5
$\left(z^2 - 2 \cdot 5 \cdot z\right) + 5^2 = (-5) + 5^2$	because	add square of 10/2
$(z-5)^2 = 20$	because	"square was completed" by previous step
$\left\lvert z - 5 \right\rvert = \sqrt{20}$	because	apply square root function, use $\sqrt{w^2} = \lvert w \rvert$

Finish by noting that $|w| = c$ is equivalent to $w = \pm\, c$. The solutions of the original equation are $z_1 = 5 + \sqrt{20}$ and $z_2 = 5 - \sqrt{20}$.

(b) Quadratic formula applied to $-x^2 + 3\,x + 9 = 0$ implies

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot (-1) \cdot (9)}}{2 \cdot (-1)}$$
$$= \frac{-3 \pm \sqrt{9 + 36}}{-2}$$
$$= \frac{-3 \pm \sqrt{45}}{-2}$$
$$= \frac{3}{2} \mp \frac{3\sqrt{5}}{2}$$
$$\approx 1.5 \mp 3.354$$

Decimal approximations to those solutions are $x_1 \approx -1.854$ and $x_2 \approx 4.854$.